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Properties of a bidisperse particle–gas suspension Part 1. Collision time small compared with viscous relaxation time

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The properties of a dilute bidisperse particle–gas suspension under low Reynolds number, high Stokes number conditions are studied in the limit $\tau_c \ll \tau_v$ using a perturbation analysis in the small parameter υ, which is proportional to the ratio of timescales τ_c/τ_v. Here, τ_c is the time between successive collisions of a particle, and τ_v is the viscous relaxation time. The leading-order distribution functions for the two species are isotropic Gaussian distributions, and are identical to the molecular velocity distributions in a two-component gas at equilibrium. Balance equations are written for the mean and mean-square velocities, using a distribution function that is a small perturbation from the isotropic Gaussian. The collisional terms are calculated by performing an ensemble average over the relative configurations of the colliding particles, and the mean velocity and velocity variances are calculated correct to $O(\upsilon^2)$ by solving the balance equations. The difference in the mean velocities of the two species is $O(\upsilon)$ smaller than the mean velocity of the suspension, and the fluctuating velocity is $O(\upsilon^{\frac{1}{2}})$ smaller than the mean velocity.

1. Introduction

Particle–gas suspensions are found in naturally occurring situations, such as dust and aerosol particles suspended in air, as well as in industrial applications, such as fluidized beds and pneumatic transport. The dynamics of these suspensions are influenced by particle inertia, gas inertia and viscosity, and hydrodynamic and collisional interactions between the particles. Owing to the complexity of the system, it is difficult, in general, to calculate the distribution of particle velocities, and continuum theories have been used to describe the dynamics of fluidized beds. These theories treat the particle and gas phases as two continuous phases capable of exchanging momentum and energy.

In his stability analysis of fluidized beds, Jackson (1963) included particle inertia, and assumed that the drag force is of the form $\boldsymbol{D} = \beta(n)\ (\boldsymbol{u}-\boldsymbol{v})$. Here, $\beta(n)$ is a function of number density, $\boldsymbol{u}$ is the mean fluid velocity and $\boldsymbol{v}$ is the mean velocity of the particle phase. This theory led to the conclusion that the homogeneous state of the fluidized bed is always unstable. More recent continuum theories (Didwania & Homsy 1982; Batchelor 1988) incoporate the particle interactions in the form of a 'particle pressure' and a particle diffusivity and thereby obtain criteria for the stability of the homogeneous bed. In the kinetic theory of gases, the pressure is proportional to the mean-square of the fluctuating velocity of the molecules. By analogy, in particle–gas suspensions the particle pressure has been related to the mean-square of the fluctuating velocities of the particles (Koch 1990; Jenkins & Richman 1985).

In this series of papers, we study the velocity distribution of a dilute bidisperse suspension of particles settling in a gas in the low Reynolds number, high Stokes number limit. The volume fraction of the particles, V, is small compared to 1. The Reynolds number is defined as $Re = (\rho_g Ua/\eta)$, and, in the low Reynolds number limit, the viscous forces are large compared to the inertial forces in the gas phase. The Stokes number is defined as $St = (mU/(6\pi\eta a^2))$, and, in the high Stokes number limit, the inertia of the particles is significant. Here, ρ_g and η are the density and viscosity of the gas, m and a are the mass and diameter of the particle, and U is a characteristic particle velocity. Particles of density 1 gm/cm^3 having a diameter between 10 and 100 μm settling in air can be analysed using the low Reynolds number, high Stokes number approximation, since their Reynolds number varies between 8×10^{-3} and 8, and their Stokes number varies between 1.473 and 1473. Moreover, the calculation of the velocity distribution in this limit is simplified by the following approximations: (i) the particle drag is given by Stokes law, (ii) the inertia of the gas can be neglected, and (iii) the particles interact only by solid-body collisions.

The drag force on the particle is a linear function of its velocity at low Reynolds number, and is assumed to be independent of the volume fraction of the particles. It is shown in the Appendix to Kumaran & Koch (1993*a*, which will henceforth be referred to as Part 2), that the effect of hydrodynamic interactions is small compared to that of solid-body collisons in a dilute polydisperse suspension at sufficiently high Stokes number, and in this study we neglect the hydrodynamic interactions between particles. Hydrodynamic interactions play a significant role in the dynamics of monodisperse suspensions, and the stability criterion for these suspensions was ascertained by Koch (1990). The simple form of the drag force considered here makes it possible to incorporate the collisional interactions in a detailed fashion. This will help in gaining a better qualitative understanding of the effect of collisions on the velocity fluctuations in complex systems such as fluidized beds, where gas inertia and hydrodynamic interactions may also be significant.

The conservation equation for the particle velocity distribution function is similar to the Boltzmann equation used in the kinetic theory of gases, except for the important difference that the drag force on a particle depends on its velocity. The conservation equation is a nonlinear, integro-differential equation, and, in general, is difficult to solve analytically. For the special case of a gas at equilibrium, the Maxwell–Boltzmann distribution of molecular velocities can be obtained as the analytical solution of the Boltzmann equation. This distribution is derived using the principle of detailed balancing which states that, at the molecular level, for every collision that changes a particle velocity from $\boldsymbol{v}$ to $\boldsymbol{v}'$, there is an 'inverse collision' which changes the velocity of another particle from $\boldsymbol{v}'$ to $\boldsymbol{v}$. Therefore, the collision process does not change the density of particles in velocity space. It is shown in the Appendix that the principle of detailed balancing is not valid for the particle velocity distribution at steady state, since the viscous drag force is not divergence free in velocity space. As a result, it is difficult to obtain an analytical solution for the distribution function, except in asymptotic limits.

The steady state of a bidisperse suspension differs from a two-component gas at equilibrium in one other respect. In a gas at equilibrium, the total energy of the molecular fluctuations is conserved. In a bidisperse suspension, on the other hand, there is a force on the particles due to the difference between the mean velocity and the terminal velocity of the particles. The work done by this force acts as a source of fluctuating energy. This source is balanced by the energy dissipation due to the viscous drag on the particles. Thus, there is a flux of energy through the system at steady state. The energy source is driven by the difference in terminal velocities in the vertical direction, and due to the directional nature of this source, the steady-state velocity distribution can be anisotropic. This is in contrast to the kinetic theory for gases near equilibrium, where the temperature, which is proportional to the mean-square fluctuating velocity, is isotropic, even though there may be directional temperature gradients.

Particle collisions have been incorporated into the theories for the rapid shearing of granular materials (see, for example, Jenkins 1987). Here, the shearing of the material drives collisions between particles and acts as a source of fluctuating energy. The energy is dissipated due to inelastic collisions between particles. The suspension is analysed in the limit where the coefficient of restitution of the particles is close to 1. In the limit, the dynamics of the particles resembles that of the molecules in a gas, whose equilibrium distribution is the Maxwell–Boltzmann distribution. The particle distribution is assumed to be an anisotropic Gaussian function, which is a small perturbation to the Maxwell distribution. The momentum and energy balance equations are derived, using this distribution, by averaging methods similar to those used in the kinetic theory of gases.

In Parts 1 and 2, we calculate the velocity distribution function for particles settling in a gas in two asymptotic limits which are defined by the relative magnitudes of two timescales: (i) the viscous relaxation time, τ_v, which is the time it takes a particle to relax to its terminal velocity after a collision, and (ii) the collision time, τ_c, which is the time between successive collisions of a particle.

The limit $\tau_c \ll \tau_v$, which corresponds to $St\,V \gg 1$, is studied in this paper. We use a perturbation analysis in the small parameter υ, which is proportional to the ratio of timescales, τ_c/τ_v. It will be shown in §2 that the small parameter υ is also proportional to $(St\,V)^{-\frac{2}{3}}$. In this limit, a particle does not experience significant viscous deceleration between successive collisions. The deceleration can be neglected in the leading-order approximation, and it is shown that the leading-order distribution functions for the two species are the Maxwell–Boltzmann distribution with equal mean velocities. To calculate the moments of the velocity distribution, however, we need to take into account the small effects of the viscous drag. This is in contrast to a gas at equilibrium, where the temperature is specified as a thermodynamic property. Balance equations are derived using a slightly perturbed form of the distribution function, and these are solved to give the mean and mean-square velocities. The collisional terms in the balance equations are calculated using an ensemble averaging method discussed in §2.4. The analysis in §2 is restricted to suspensions of elastic particles, and suspensions of inelastic particles are studied in §3.

In Part 2, the limit $\tau_v \ll \tau_c$, which corresponds to the limit $St\,V \ll 1$, will be analysed. In this limit, a particle relaxes to its terminal velocity between successive collisions. A distribution function that incorporates the first effects of collisions is derived by a perturbation analysis about the base state in which all the particles settle at their terminal velocities. In Kumaran & Koch (1993*b*), we calculate the mean and mean-square velocities for values of $St\,V$ between the two limits using an approximate form of the distribution function.

2. Suspension of elastic particles

2.1. *Velocity scales*

The system consists of particles of two species, 1 and 2, with masses m_1 and m_2, radii a_1 and a_2, and number densities n_1 and n_2, respectively, settling under gravity in a quiescent gas. The drag force on a particle of species i is given by

$$F_i^\dagger = -\mu_i v_i^\dagger. \tag{2.1}$$

Here $v_i^\dagger$ is the particle velocity and μ_i is $6\pi\eta a_i$. The superscript $\dagger$ is used for dimensional velocity and time variables, and the absence of the superscript indicates scaled variables.

There are two mechanisms of energy transfer that influence the dynamics: the collisional mechanism, which channels energy from the mean flow into the velocity fluctuations, and the viscous mechanism, which dissipates the fluctuating energy. The rate of change of energy due to the collisional mechanism is inversely related to the collision time scale, τ_{cij}, which is the time between successive collisions of a particle of species i with particles of species j. The collision time is related to the radius, number density and fluctuating velocity of the particles as follows:

$$\tau_{cij} = 1/[n_j \pi d_{ij}^2 v_f^\dagger]. \tag{2.2}$$

Here, $v_f^\dagger$ is the magnitude of the fluctuating velocity of the particles and d_{ij} is the sum of the radii of particles of species i and j. The rate of dissipation of energy is inversely related to the viscous relaxation time τ_{vi}, which is the time it takes for a particle to relax to its terminal velocity after a collision. From (2.1), the viscous relaxation time is given by

$$\tau_{vi} = m_i/\mu_i. \tag{2.3}$$

To facilitate the perturbation analysis, we define a small parameter, υ, as the ratio of the collision and viscious timescales:

$$\upsilon = \tau_{c12}/\tau_{v1}. \tag{2.4}$$

Note that the choice of the viscious relaxation time for species 1 and the collision time for collisions between particles of species 1 and 2 is arbitrary. Since the timescales of the two species are of the same order of magnitude, this choice does not affect the scalings in the problem. It is shown in the subsequent analysis that the mean velocities of the two species are equal to leading order, and the fluctuating velocity $v_f^\dagger$ is $O(\upsilon^{\frac{1}{2}} v_m^\dagger)$, where $v_m^\dagger$ is the mean velocity of the suspension. Therefore, in terms of the mean velocity of the supension, υ can be expressed as

$$\upsilon = [n_2 \pi d_{12}^2 v_m^\dagger \tau_{v1}]^{-\frac{2}{3}}. \tag{2.5}$$

It can easily be verified from (2.5) that υ is proportional to $(St\, V)^{-\frac{2}{3}}$, as indicated in the introduction. Note that the above scaling for the fluctuating velocity is valid only when $(\upsilon^{\frac{1}{2}} v_m^\dagger)$ is small compared with the difference in the terminal velocity of the two species. When the difference in the terminal velocities is small compared with $(\upsilon^{\frac{1}{2}} v_m^\dagger)$, the fluctuating velocity will scale as the difference in the terminal velocities. The latter limit is analysed in Part 2.

We shall now estimate the orders of magnitude of the difference in the mean velocities of the two species and the mean-square fluctuating velocities in terms of the small parameter υ. The order of magnitude of the mean velocity at steady state is estimated by balancing the viscous and collisional rate of change of momentum in the suspension. The force on a particle of species i due to viscous drag is $O[m_i v_{im}^\dagger/\tau_{vi}]$, and the rate of change of momentum of the particle due to collisions is $O[m_i\,(v^\dagger_{1\mathrm{m}}-v^\dagger_{2\mathrm{m}})/\tau_{cij}]$. Here, $v^\dagger_{i\mathrm{m}}$ is the mean velocity of species i, and the difference in the mean velocities, $v^\dagger_{1\mathrm{m}}-v^\dagger_{2\mathrm{m}}$, drives the momentum transfer. At steady state these two rates are equal, and the ratio $(v^\dagger_{1\mathrm{m}}-v^\dagger_{2\mathrm{m}})/v^\dagger_{i\mathrm{m}}$ is $O(\upsilon)$. The leading-order mean velocities of the two species are equal to the mean velocity of the suspension, $v^\dagger_{\mathrm{m}}$, and the difference in the mean velocities is $O(\upsilon)$ smaller than this mean velocity. Since momentum is conserved in collisions between particles, the mean velocity of the suspension at steady state can be determined from the condition that the sum of the gravitational and drag forces acting on all the particles is zero:

$$v^\dagger_{\mathrm{m}} = \frac{n_1\,\mu_1\,v^\dagger_{1\mathrm{t}}+n_2\,\mu_2\,v^\dagger_{2\mathrm{t}}}{n_1\,\mu_1+n_2\,\mu_2}, \tag{2.6}$$

where $v^\dagger_{1\mathrm{t}}$ and $v^\dagger_{2\mathrm{t}}$ are the terminal velocities of the two species. The mean velocities of the two species are expressed as perturbation series about the mean velocity of the suspension:

$$v^\dagger_{i\mathrm{m}} = v^\dagger_{\mathrm{m}}\,(1+\upsilon v'_i). \tag{2.7}$$

The fluctuating velocity of the particles, $\boldsymbol{c}^\dagger_i$, is defined as difference between the particle velocity $\boldsymbol{v}^\dagger_i$ and the mean velocity of the suspension:

$$\boldsymbol{c}^\dagger_i = \boldsymbol{v}^\dagger_i - v^\dagger_{\mathrm{m}}\,\boldsymbol{e}_z. \tag{2.8}$$

The order of magnitude of the fluctuating velocity is determined from the energy conservation equation. The leading-order vertical momentum conservation and total energy conservation equations are

$$n_1\,\mu_1\,(v^\dagger_{1\mathrm{t}}-v^\dagger_{\mathrm{m}})+n_2\,\mu_2\,(v^\dagger_{2\mathrm{t}}-v^\dagger_{\mathrm{m}}) = 0, \tag{2.9a}$$

$$n_1\,\mu_1\,v^\dagger_{\mathrm{m}}(v^\dagger_{1\mathrm{t}}-v^\dagger_{\mathrm{m}})+n_2\,\mu_2\,v^\dagger_{\mathrm{m}}(v^\dagger_{2\mathrm{t}}-v^\dagger_{\mathrm{m}})-n_1\,\mu_1\,\langle c^{\dagger 2}_1\rangle-n_2\,\mu_2\,\langle c^{\dagger 2}_2\rangle = 0. \tag{2.9b}$$

The sum of the first two terms in the energy conservation equation (2.9*b*) is the product of the momentum conservation equation (2.9*a*) and the mean velocity of the suspension, $v^\dagger_{\mathrm{m}}$. Therefore, the mean-square fluctuating velocities are much smaller than $v^{\dagger 2}_{\mathrm{m}}$, and the leading-order fluctuating velocity is $O(\upsilon^{\frac{1}{2}}v^\dagger_{\mathrm{m}})$. This confirms the scaling of the fluctuating velocity anticipated earlier in this section.

In the subsequent analysis, the particle velocities are scaled by the fluctuating velocity $v^\dagger_{\mathrm{f}} = \upsilon^{\frac{1}{2}}v^\dagger_{\mathrm{m}}$, and the time variable is scaled by $\tau_{\mathrm{v}1}$, the viscous relaxation time of species 1. The mean and terminal velocities are expressed as

$$v^\dagger_{i\mathrm{m}} = \upsilon^{-\frac{1}{2}}v^\dagger_{\mathrm{f}}(1+\upsilon v'_i), \qquad v^\dagger_{i\mathrm{t}} = \upsilon^{-\frac{1}{2}}v^\dagger_{\mathrm{f}}\,v_{i\mathrm{t}}. \tag{2.10a,b}$$

Here v'_i and v_{it} are dimensionless $O(1)$ quantities. The scaled acceleration of a particle due to the viscous and gravitational forces is

$$\mathrm{d}\boldsymbol{c}_i/\mathrm{d}t = [\upsilon^{-\frac{1}{2}}(v_{i\mathrm{t}}-1)\boldsymbol{e}_z-\boldsymbol{c}_i](\tau_{\mathrm{v}1}/\tau_{\mathrm{v}i}). \tag{2.11}$$

2.2. *Particle velocity distribution*

In this section, we calculate the distribution of velocities of the particles in the suspension. The dimensional velocity distribution function is defined as follows: $f^\dagger_i(\boldsymbol{c}^\dagger_i)\,\mathrm{d}\boldsymbol{c}^\dagger_i$ is the fraction of the particles of species i that have velocities in the differential volume $\mathrm{d}\boldsymbol{c}^\dagger_i$ about $\boldsymbol{c}^\dagger_i$ in velocity space. The ensemble average of a function of particle velocity, $\beta_i(\boldsymbol{c}^\dagger_i)$, is

$$\langle\beta_i\rangle = \int_{\boldsymbol{c}^\dagger_i}\beta_i(\boldsymbol{c}^\dagger_i)\,f_i(\boldsymbol{c}^\dagger_i)\,\mathrm{d}\boldsymbol{c}^\dagger_i \tag{2.12}$$

The conservation equation for the distribution function is

$$\frac{\partial f_i^\dagger}{\partial t^\dagger} = -\nabla_{c_i^\dagger}\cdot\left[\frac{\mathrm{d}\boldsymbol{c}_i^\dagger}{\mathrm{d}t^\dagger} f_i^\dagger(\boldsymbol{c}_i^\dagger)\right] + \frac{\partial_c f_i^\dagger}{\partial t^\dagger}. \tag{2.13}$$

Here $\nabla_{c_i^\dagger}\cdot$ is the divergence operator in velocity space, $\partial_c f_i^\dagger/\partial t^\dagger$ is the net accumulation of particles in $\mathrm{d}\boldsymbol{c}_i^\dagger$ due to collisions, and the first term on the right-hand side is the accumulation of particles in $\mathrm{d}\boldsymbol{c}_i^\dagger$ due to the viscous and gravitational forces. In the Boltzmann equation, the forces on the gas molecules are divergence free in velocity space, but in (2.13) the forces are not divergence free and the acceleration must be written within the divergence operator. The implications of this difference are discussed in the Appendix.

Since the viscous relaxation time is large compared to the collision time, a particle does not decelerate much between successive collisions. The leading-order dynamics of the particles is similar to that of molecules in an ideal gas in the absence of external forces. This correspondence can be better illustrated by scaling the conservation equation (2.13). The velocity coordinate is scaled by $v_f^\dagger$, the fluctuating velocity. The collision integral scales as the collision frequency multiplied by the order of magnitude of the distribution function:

$$\partial_c f_i^\dagger/\partial t^\dagger = O(f_i^\dagger/\tau_{cij}). \tag{2.14}$$

The viscous accumulation term scales as

$$\nabla_{c_i^\dagger}\cdot\left[\frac{\mathrm{d}\boldsymbol{c}_i^\dagger}{\mathrm{d}t^\dagger} f_i^\dagger(\boldsymbol{c}_i^\dagger)\right] = O\left(\frac{f_i^\dagger}{\upsilon^{\frac{1}{2}}\tau_{vi}}\right). \tag{2.15}$$

There is a factor of $\upsilon^{\frac{1}{2}}$ in the denominator because the divergence operator in velocity space scales as $(1/v_f^\dagger)$, while the viscous deceleration, which is proportional to the difference between the particle velocity and its terminal velocity, scales as $v_m^\dagger/\tau_{vi}$.

From (2.14) and (2.15), it can be seen that the viscous term is $O(\upsilon^{\frac{1}{2}})$ smaller than the collisional term, and the collision integral is zero to leading order at steady state. The molecular distribution function for a gas at equilibrium is also derived from the zero collision integral condition (see Chapman & Cowling 1970, chap. 3), and to describe the leading-order dynamics of the suspension we can make use of two results derived for multi-component gas mixtures:

(i) The velocity distribution function of species i is an isotropic Gaussian function of the fluctuating velocity $\boldsymbol{c}_i$:

$$f_i = \exp(-c_i^2/\xi_i)/[\pi\xi_i]^{\frac{3}{2}}. \tag{2.16}$$

Here, ξ_i is the variance of the Gaussian distribution. The above equation is identical to the Maxwell–Boltzmann distribution for the molecules of a gas at equilibrium, in which the velocity variance, ξ_i, is expressed as $2kT/m_i$, where T is the gas temperature, m_i is the mass of a molecule of component i, and k is the Boltzmann constant.

(ii) In a gas at equilibrium, the temperatures of the components are equal. By analogy, the variances of the two species in the bidisperse suspension are related by

$$m_1\xi_1 = m_2\xi_2. \tag{2.17}$$

In a gas, the temperature is specified as a thermodynamic property, but in the suspension it is necessary to take into account the small perturbation caused by viscous forces to calculate the variances. The exact form of the perturbation is difficult to calculate analytically, and numerical methods have been developed to calculate the distribution function for gases that are far from equilibrium (Yen 1984).

For systems whose velocity distributions are close to the Maxwell–Boltzmann distribution, the perturbation is assumed to be in the form of a polynomial expansion in the fluctuating velocities. A perturbation in the form of a Hermite polynomial expansion is used in the theory of granular flows (Jenkins 1987). Hermite polynomials constitute an orthogonal function space if the inner product is defined with a Gaussian weighting function, and a Hermite polynomial expansion simplifies the calculation of the moments of the distribution. Keeping all the polynomials gives the exact form of the perturbation for small deviations from equilibrium, but typically only a few are kept owing to the complexity of the algebra involved.

In this analysis, the distribution function for species i is assumed to be an anisotropic Gaussian distribution about its mean velocity:

$$f_i = \frac{1}{(\pi\xi_{ir})(\pi\xi_{iz})^{\frac{1}{2}}}\exp\left[-\left(\frac{(c_{iz}-\upsilon^{\frac{1}{2}}v'_{im})^2}{\xi_{iz}}+\frac{c_{ir}^2}{\xi_{ir}}\right)\right]. \tag{2.18}$$

The variances in the horizontal and vertical directions are small perturbations to the isotropic variance, ξ_i, as indicated by (2.16):

$$\xi_{ir} = \xi_i + \upsilon\xi'_{ir}, \quad \xi_{iz} = \xi_i + \upsilon\xi'_{iz}. \tag{2.19a, b}$$

If we substitute the above expressions for the velocity variances into (2.18), expand in a Taylor series about $\upsilon = 0$, and retain terms up to $O(\upsilon)$, we get the following expression for the velocity distribution function:

$$f_i = \frac{1}{(\pi\xi_i)^{\frac{3}{2}}}\exp\left(-\frac{c_i^2}{\xi_i}\right)\left\{1+2\upsilon^{\frac{1}{2}}\frac{c_{iz}v'_{im}}{\xi i}+\upsilon\left[\frac{c_{iz}^2}{\xi_i}\left(\frac{2v'^2_{im}}{\xi_i}+\frac{\xi'_{iz}}{\xi_i}\right)+\frac{c_{ir}^2\,\xi'_{ir}}{\xi_i^2}-\frac{v'^2_{im}}{\xi_i}-\left(\frac{\xi'_{iz}+2\xi'_{ir}}{2\xi_i}\right)\right]\right\}. \tag{2.20}$$

The above expression is identical to a Hermite polynomial expansion that includes the linear and quadratic terms. We use the form of the distribution function in (2.18) because it simplifies the calculation of the collisional terms by the ensemble averaging method developed in §2.4. The parameters in the distribution function are calculated from the balance equations for the moments of the distribution.

2.3. *Balance equations*

Balance equations for the following three moments of the fluctuating velocity are used to calculate the parameters in the distribution function (2.20): (1) mean velocity in the vertical direction, $\langle c_{iz}\rangle$; (ii) mean-square of the fluctuating velocity, $\langle \boldsymbol{c}_i.\boldsymbol{c}_i\rangle$; (iii) mean-square of the horizontal component of the fluctuating velocity, $\langle \boldsymbol{c}_{ir}.\boldsymbol{c}_{ir}\rangle$. The subscript r will henceforth be used to denote the projection of the velocity vector in the horizontal plane. The balance equations are derived by multiplying the conservation equation (2.13) by the velocities and the square of the velocities, and integrating over the velocity domain of species i. This procedure is described in detail in Chapman & Cowling (1970, Chap. 3). The leading-order balance equations at steady state for the moments are

$$-\left(\frac{\tau_{v1}}{\tau_{vi}}\right)[\upsilon^{-\frac{1}{2}}(v_{it}-1)]+\frac{\partial_c\langle c_{iz}\rangle}{\partial t} = 0, \tag{2.21a}$$

$$-2\left(\frac{\tau_{v1}}{\tau_{vi}}\right)[v'_{im}(v_{it}-2)-\langle c_i^2\rangle]+\frac{\partial_c\langle c_i^2\rangle}{\partial t} = 0, \tag{2.21b}$$

$$-2\left(\frac{\tau_{v1}}{\tau_{vi}}\right)\langle c_{ir}^2\rangle+\frac{\partial_c\langle c_{ir}^2\rangle}{\partial t} = 0. \tag{2.21c}$$

In the above equations, $(\partial_c/\partial t)$ is the change of the velocity moments due to collisons between particles. There is a factor of $v^{-\frac{1}{2}}$ in the first term on the right-hand side of (2.21 *a*) due to the scaling of the mean velocity and terminal velocities (see (2.8) and (2.9)). The mean-square fluctuating velocities, $\langle c_i^2 \rangle$ and $\langle c_{ir}^2 \rangle$, are related to the variances of the distribution function in the horizontal and vertical directions, ξ_{ir} and ξ_{iz}, as follows:

$$\langle c_{ir}^2 \rangle = \xi_{ir}, \quad \langle c_i^2 \rangle = \xi_{ir} + \tfrac{1}{2}\xi_{iz}. \qquad (2.22\,a, b)$$

Since collisions between particles are elastic and conserve momentum and energy, the collisional rates of change of mean and mean-square velocities satisfy the conditions

$$n_1 m_1 \frac{\partial_c \langle c_{1z} \rangle}{\partial t} + n_2 m_2 \frac{\partial_c \langle c_{2z} \rangle}{\partial t} = 0, \qquad (2.23)$$

$$n_1 m_1 \frac{\partial_c \langle c_1^2 \rangle}{\partial t} + n_2 m_2 \frac{\partial_c \langle c_2^2 \rangle}{\partial t} = 0. \qquad (2.24)$$

The momentum balance equation (2.21 *a*) is identical to that used in the kinetic theory for a two-component gas mixture (Tham & Gubbins 1971), and in the analysis of granular flows (Jenkins & Mancini 1989). However, the treatment of the energy balance equation is different from that used in mixture theory in the following respects: (i) Usually just one energy conservation equation is written for the average temperature of the gas in mixture theory, whereas here, balance equations are written for the mean-square velocities for each species in the horizontal and vertical directions. This allows us to take into account the anisotropy in the distribution, and to calculate the $O(v)$ corrections to the velocity variances. (ii) In mixture theory, the work done due to the external forces is given by $\boldsymbol{F}_i \cdot \boldsymbol{v}_i'$, where $\boldsymbol{F}_i$ is the external force, and $\boldsymbol{v}_i'$, the 'diffusion velocity' of species i, is vv'_{im} in our case. In the analysis of Jenkins & Mancini (1989) for example, the external work is done by an external shear force acting on the system, which is independent of the 'diffusion velocity'. In our system, the leading-order force on the suspension is proportional to $v_{it}-1$, the difference between the terminal velocity and the mean velocity of the suspension. However, it can be seen from (2.21 *b*) that the work done by the drag force is proportional to $vv'_{im}(v_{it}-2)$, which is not the product of the external force and the diffusion velocity. Owing to the velocity dependence of the force, there is an additional contribution to the work done on the particles which is proportional to the product of the $O(v)$ correction to the force and the mean velocity of the suspension. Therefore, the leading-order non-trivial energy balance equation for the present case is different from that used in mixture theory.

2.4. *Collisional change in the velocity moments*

In this section, we use ensemble averaging to derive expressions for the collisional rate of change of the velocity moments in a uniform suspension. Consider two particles of species i and j (i and j can be 1 or 2) whose velocities are in the volumes $\mathrm{d}\boldsymbol{c}_i$ about $\boldsymbol{c}_i$ and $\mathrm{d}\boldsymbol{c}_j$ about $\boldsymbol{c}_j$ respectively. The velocities of the particles are expressed in terms of the velocity of the centre of mass of the two particles (the common velocity $\boldsymbol{q}$) and the difference in velocity between them ($\boldsymbol{w}$):

$$\boldsymbol{q} = \frac{m_i \boldsymbol{c}_i + m_j \boldsymbol{c}_j}{m_i + m_j}, \qquad \boldsymbol{w} = \boldsymbol{c}_i - \boldsymbol{c}_j. \qquad (2.25\,a, b)$$

At this point, it is necessary to define a number of coordinate systems to characterize the collision. The spherical coordinate system $C0$ has its origin at the

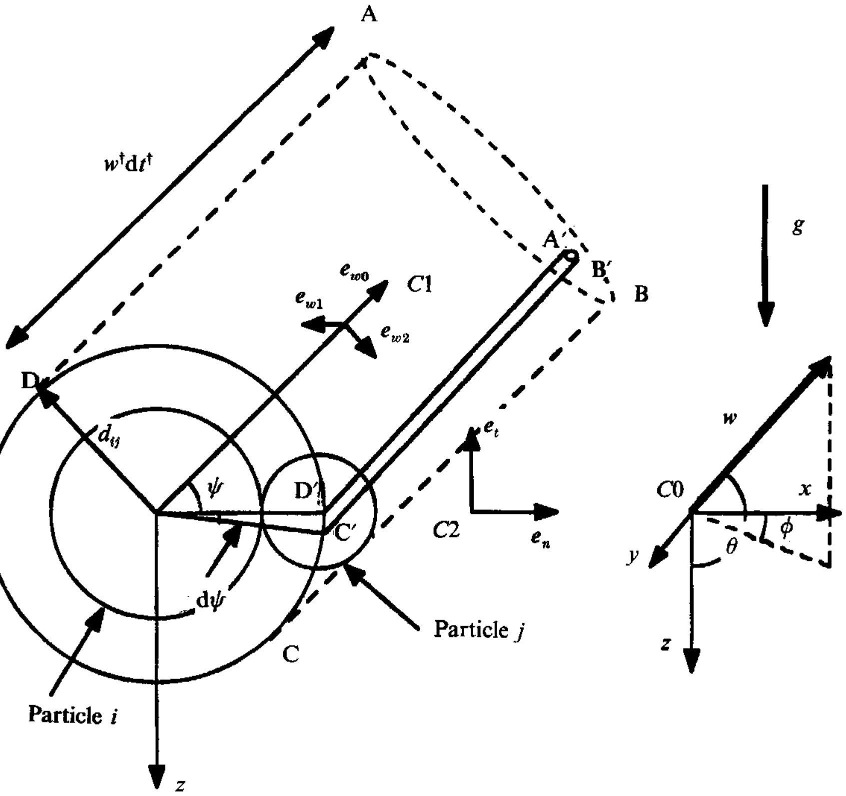

FIGURE 1. Coordinate systems for calculating collisional rate of change of mean and mean-square velocities.

centre of the particle of species i and its axis in the direction of gravity, as shown in figure 1. The magnitude of the relative velocity is in the interval dw about w, and the relative velocity vector is in the differential solid angle $\sin\theta\, d\theta\, d\phi$ about the orientation (θ, ϕ). Here θ is the azimuthal angle and ϕ is the meridional angle in $C0$. The unit vector in the direction of the relative velocity, $\boldsymbol{e}_{w0}$, and the two unit vectors perpendicular to it, $\boldsymbol{e}_{w1}$ and $\boldsymbol{e}_{w2}$, form a Cartesian coordinate system $C1$. The relations between the unit vectors in the coordinate systems $C0$ and $C1$ are

$$\boldsymbol{e}_{w0} = \cos\theta\, \boldsymbol{e}_z + \sin\theta \cos\phi\, \boldsymbol{e}_x + \sin\theta \sin\phi\, \boldsymbol{e}_y. \tag{2.26a}$$

$$\boldsymbol{e}_{w1} = \sin\phi\, \boldsymbol{e}_x - \cos\phi\, \boldsymbol{e}_y, \tag{2.26b}$$

$$\boldsymbol{e}_{w2} = -\sin\theta\, \boldsymbol{e}_z + \cos\theta \cos\phi\, \boldsymbol{e}_x + \cos\theta \sin\phi\, \boldsymbol{e}_y. \tag{2.26c}$$

The particles collide in a time $dt^\dagger$ if the centre of the second particle is in the cylinder ABCD, whose volume is given by $\pi d_{ij}^2 w^\dagger dt^\dagger$. Consider a collision in which the impact vector, which is the vector joining the centres of the particles at the point of collision, is in the differential solid angle $\sin\psi\, d\psi\, d\eta$ about ψ, η relative to the direction of $\boldsymbol{w}$. Here ψ is the azimuthal angle in the coordinate system $C1$ which varies from 0 to $\frac{1}{2}\pi$, and η is the meridional angle which varies from 0 to 2π. The unit vector $\boldsymbol{e}_n$ is normal to the surface at the point of collision, and $\boldsymbol{e}_t$ is the tangent to the surface in the plane of the relative velocity, $\boldsymbol{w}$, and the normal, $\boldsymbol{e}_n$. The unit vectors $\boldsymbol{e}_n$ and $\boldsymbol{e}_t$ are related to the unit vectors in $C1$ as follows:

$$\boldsymbol{e}_n = \cos\psi\, \boldsymbol{e}_{w0} + \sin\psi \cos\eta\, \boldsymbol{e}_{w1} + \sin\psi \sin\eta\, \boldsymbol{e}_{w2}, \tag{2.27a}$$

$$\boldsymbol{e}_t = \sin\psi\, \boldsymbol{e}_{w0} - \cos\psi \cos\eta\, \boldsymbol{e}_{w1} - \cos\psi \sin\eta\, \boldsymbol{e}_{w2}. \tag{2.27b}$$

The particles collide with this impact vector in a time $dt^\dagger$ if the centre of the second particle is the region A′B′C′D′, whose volume is $w^\dagger d_{ij}^2 \cos\psi \sin\psi\, d\psi\, d\eta\, dt^\dagger$. The number of collisions between the particles in this specified orientation per unit volume in a time $dt^\dagger$ is the product of the number densities of the particles having velocities $\boldsymbol{c}_i$ and $\boldsymbol{c}_j$ and the volume of the region A′B′C′D′:

$$\frac{\text{no. of collisions}}{\text{volume}} = n_i f_i(\boldsymbol{c}_i)\, n_j f_j(\boldsymbol{c}_j)\,(w^\dagger \mathrm{d}_{ij}^2 \cos\psi)\sin\psi\,\mathrm{d}\psi\,\mathrm{d}\eta\,\mathrm{d}\boldsymbol{c}_i\,\mathrm{d}\boldsymbol{c}_j\,\mathrm{d}t^\dagger. \quad (2.28)$$

The collision frequency of a particle of species i is the ratio of (2.15) and the number density n_i. Using (2.2) for the collision time, τ_{cij}, the collision frequency, scaled by τ_{v1}, is

$$\text{frequency of collisions} = v\left[\frac{\tau_{c12}}{\pi\tau_{cij}}\right] f_i(\boldsymbol{c}_i) f_j(c_j)\,\omega\cos\psi\sin\psi\,\mathrm{d}\psi\,\mathrm{d}\eta\,\mathrm{d}\boldsymbol{c}_i\,\mathrm{d}\boldsymbol{c}_j. \quad (2.29)$$

In deriving (2.29) we have assumed that the pair probability distribution function for two colliding particles is a product of the single particle velocity distribution functions. This assumption is valid when the volume fraction of the particles is low, and the velocities of colliding particles are not correlated.

In a collision, the velocity of the centre of the mass remains unchanged, and the difference velocities before and after collision are related by

$$\boldsymbol{w} = w\,\boldsymbol{e}_{w0} = w(\cos\psi\,\boldsymbol{e}_n + \sin\psi\,\boldsymbol{e}_t), \quad (2.30a)$$

$$\boldsymbol{w}^* = w(-e\cos\psi\,\boldsymbol{e}_n + \sin\psi\,\boldsymbol{e}_t). \quad (2.30b)$$

The superscript * is used to denote velocities after collision. In (2.30*b*), e is the coefficient of restitution. The analysis in §2 is confined to systems of elastic particles, for which e is 1, and suspensions of inelastic particles are examined in §3.

From (2.30), the changes in the velocity moments of a particle of species i due to a collision with a particle of species j are

$$\Delta_{ij}(c_{iz}) = \frac{m_j}{m_i+m_j}(w_z^* - w_z), \quad \Delta_{ij}(c_i^2) = 2\frac{m_j}{m_i+m_j}[\boldsymbol{q}\cdot(\boldsymbol{w}^* - \boldsymbol{w})], \quad (2.31a,b)$$

$$\Delta_{ij}(c_{ir}^2) = 2\frac{m_j}{m_i+m_j}\boldsymbol{q}_r\cdot(\boldsymbol{w}_r^* - \boldsymbol{w}_r) + \left(\frac{m_j}{m_i+m_j}\right)^2 (w_r^{*2} - w_r^2). \quad (2.31c)$$

Here $\Delta_{ij}(\beta_i)$ is the change in the value of the property β_i of a particle of species i due to a collision with a particle of species j.

The average rate of change of the property, $\beta_i(\boldsymbol{c}_i)$, of species i, due to collisions with particles of species j, is evaluated by multiplying the rate of change of the property per collision (2.31) by the frequency of collisions between particles of species i and j (2.29), and integrating this over the velocity space of species i and j and over the orientation space of the impact vector:

$$\frac{\partial_c\langle\beta_i\rangle}{\partial t} = \sum_{j=1}^{2}\frac{1}{\pi\tau_{cij}}\int_{c_i}\int_{c_j}\int_{\eta=0}^{2\pi}\int_{\psi=0}^{\pi/2}\Delta_{ij}(\beta_i) f_i(\boldsymbol{c}_i) f_j(\boldsymbol{c}_j) w\cos\psi\sin\psi\,\mathrm{d}\psi\,\mathrm{d}\eta\,\mathrm{d}\boldsymbol{c}_j\,\mathrm{d}\boldsymbol{c}_i. \quad (2.32)$$

It can easily be verified that, for an isotropic Gaussian distribution in which the variances of the two species are related by (2.17), the rates of change of all the velocity moments are zero. For the perturbed Gaussian distribution (2.18), however, there is a transfer of momentum and energy between the two species due to the differences in their mean velocities and velocity variances. This is calculated by expressing the pair distribution function $f_i(\boldsymbol{c}_i) f_j(\boldsymbol{c}_j)$ in terms of the common and difference velocities (2.25), expanding it in a Taylor series about $v = 0$, and carrying out the integrations in (2.32). Note that the Jacobian for the transformation from $(\boldsymbol{c}_i, \boldsymbol{c}_j)$ to $(\boldsymbol{q}, \boldsymbol{w})$ coordinates is 1.

There is a change in the mean velocity $\langle c_{iz} \rangle$ due to the transfer of momentum to particles of species i in collisions with particles of the other species, $3-i$, which is given by

$$\frac{\partial_c \langle c_{iz} \rangle}{\partial t} = -v^{-\frac{1}{2}} \left(\frac{\tau_{c12}}{\tau_{cik}} \right) \frac{8(\xi_1 + \xi_2)^{\frac{1}{2}}}{3\pi^{\frac{1}{2}}} \frac{m_k}{m_i + m_k} (v'_{im} - v'_{km}). \tag{2.33}$$

Here, $k = 3-i$, and the factor τ_{c12}/τ_{cik} appears because of our choice of timescales for the parameter v. The collisional transfer of momentum is driven by the difference in the mean velocities of the two species, $v'_{1m} - v'_{2m}$, and is directed from the faster to the slower species. The printed variables refer to the perturbations to the velocity moments, as indicated in (2.17) and (2.19). The transfer rates given by (2.33) for the two species satisfy the momentum conservation condition (2.23). The factor $(\xi_1 + \xi_2)^{\frac{1}{2}}$ appears in the numerator because the collision frequency is proportional to the difference velocity w.

There is a change in the mean-square fluctuating velocity of species i, $\langle c_i^2 \rangle$, due to the transfer of energy in collisions with particles of the other species, $3-i$, which is given by

$$\frac{\partial_c \langle c_i^2 \rangle}{\partial t} = -\left(\frac{\tau_{c12}}{\tau_{cik}} \right) \frac{8(\xi_1 + \xi_2)^{\frac{1}{2}}}{3\pi^{\frac{1}{2}}} \frac{m_k}{m_i + m_k} \left[2\left(\frac{m_i \xi'_{ir} - m_k \xi'_{kr}}{m_i + m_k} \right) + \left(\frac{m_i \xi'_{iz} - m_k \xi'_{kz}}{m_i + m_k} \right) \right], \tag{2.34}$$

where $k = 3-i$. The transfer rates for the two species given by (2.34) satisfy the energy conservation condition (2.24). The collisional energy transfer is driven by $m_i \xi'_{iz} - m_k \xi'_{kz}$ and $m_i \xi'_{ir} - m_k \xi'_{kr}$, which are proportional to the difference between the average kinetic energies of species i and k in the vertical and horizontal directions respectively. The direction of the transfer is from the species having a higher average energy to that having a lower average energy, and tends to equalize the average energies of the particles of the two species.

The collisional rate of change of the horizontal component of the fluctuating velocity is given by

$$\frac{\partial_c \langle c_{ir}^2 \rangle}{\partial t} = \left(\frac{\tau_{c12}}{\tau_{cik}} \right) \frac{(\xi_1 + \xi_2)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \frac{m_k}{m_i + m_k} \left[\frac{16}{3} \left(\frac{m_i \xi'_{ir} - m_k \xi'_{kr}}{m_i + m_k} \right) \right.$$
$$\left. + \frac{32}{15} \frac{m_k}{m_i + m_k} [-2(v'_{im} - v'_{km})^2 + (\xi'_{iz} + \xi'_{kz} - \xi'_{ir} - \xi'_{kr})] \right] + \left(\frac{\tau_{c12}}{\tau_{cii}} \right) \frac{(2\xi_i)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \frac{8}{15} (\xi'_{iz} - \xi'_{ir}). \tag{2.35}$$

The last term in (2.35) represents the redistribution of energy in collisions between particles of the same species, and is driven by $\xi'_{iz} - \xi'_{ir}$, the difference in the velocity variances in the horizontal and vertical directions. The first term represents the change in energy due to collisions between particles of different species, and consists of energy transfers driven by three terms: (i) the difference in the average energies of the two species in the horizontal direction $m_i \xi'_{ir} - m_k \xi'_{kr}$, which tends to equalize the energies of the two species, (ii) the difference in the energies of the horizontal and vertical fluctuations $\xi'_{iz} + \xi'_{kz} - \xi'_{ir} - \xi'_{kr}$, which tends to equalize the fluctuating energies in the horizontal and vertical directions, and (iii) a contribution proportional to the square of the difference in the mean velocities.

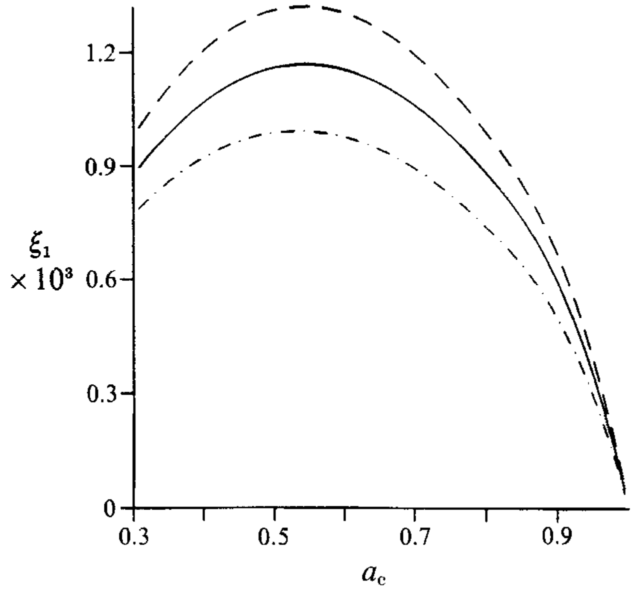

FIGURE 2. The leading-order velocity variance, ξ_1, non-dimensionalized by $v_{1t}^{\dagger 2}$, as a function of the ratio of particle sizes, $a_c = a_2/a_1$. The densities of the two species are equal and, v_1 is 0.001. - · - · - · -, $n_c = 1.5$; ——, $n_c = 1.0$; - - - - - - - - $n_c = 0.5$.

The collisional transfer rates, (2.33), (2.34) and (2.35), can be substituted into the balance equations (2.21) and solved for: (i) the leading-order velocity variance ξ_i and the $O(v)$ difference in the mean velocities v'_{1m}-v'_{2m}, (ii) the difference in the average fluctuation energies of the particles of the two species, $m_1 \xi'_{1z} - m_2 \xi'_{2z}$ and $m_1 \xi'_{1r} - m_2 \xi'_{2r}$, and (iii) the difference in the velocity variance between the horizontal and vertical fluctuations $\xi'_{iz} - \xi'_{ir}$. We have derived analytical expressions for the $O(v)$ correction to the mean velocities, v'_{im}, and the leading-order velocity variances, ξ_i:

$$v'_{im} = \left[\frac{27\pi(m_c+1)\, n_c^4 \mu_c^3(\mu_c - m_c)(n_c \mu_c + m_c)}{128 m_c^2 (m_c n_c + 1)(n_c \mu_c + m_c)^3}\right]^{\frac{1}{3}} \left(\frac{-1}{n_c \mu_c}\right)^{i-1}, \tag{2.36}$$

$$\xi_i = \left[\frac{\pi m_c (m_c+1)\, n_c^4 (\mu_c - m_c)^4}{16(m_c n_c + 1)^4 (n_c \mu_c + m_c)^2}\right]^{\frac{1}{3}} \left(\frac{1}{m_c}\right)^{i-1}. \tag{2.37}$$

Here, m_c, n_c and μ_c are m_2/m_1, n_2/n_1 and μ_2/μ_1, respectively. We have stipulated that species 1 is the heavier species, and the mean velocities and velocity variances are non-dimensionalized by $(v v_m^\dagger)$ and $(v v_m^{\dagger 2})$, respectively. The $O(v)$ corrections to the velocity variances in the vertical and horizontal directions have been calculated numerically from the higher-order balance equations.

Figures 2 and 3 show the variations in the mean-square velocities of particles of species 1 due to changes in the size ratio and the number density ratio. In these figures the dimensionless parameter, v_1, defined as $[\tau_{v1} n_1 (4\pi a_1^2) v_{1t}^\dagger]^{-\frac{2}{3}}$, is kept constant. v_1 is similar to the ratio of timescales v defined in (2.4), but is expressed in terms of the properties of species 1 to better illustrate the effect of changes in the size ratio.

Figure 2 is a plot of the velocity variance as a function of the ratio of particle sizes, a_2/a_1. The dimensionless parameter v_1 is 0.001. As the radius of particle 2 is increased relative to that of particle 1, the velocity variance first increases, since the particles of species 2 become more massive and transfer more momentum and energy to the particles of species 1. As the radius of particle 2 is further increased, however, the

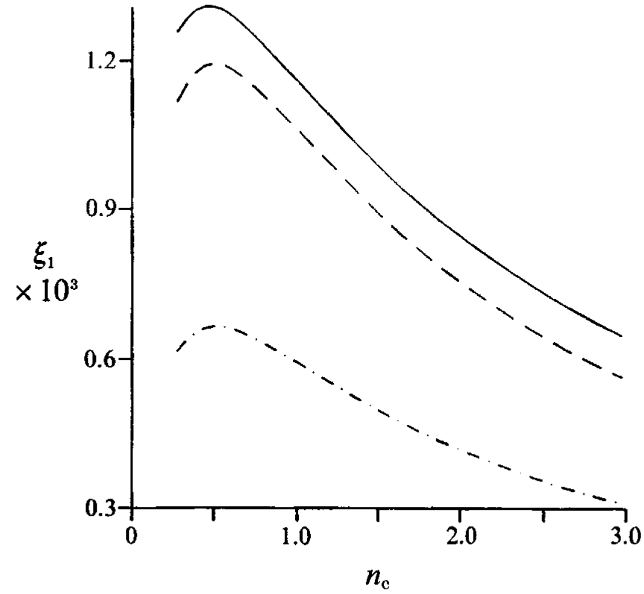

FIGURE 3. The leading-order velocity variance, ξ_1, non-dimensionalized by $v_{1t}^{\dagger 2}$, as a function of the number density ratio, $n_c = n_2/n_1$. The densities of the two species are equal, and υ_1 is 0.001. ——, $a_c = 0.5$; – – – – – – – –, $a_c = 0.7$; ·-·-·-·-·, $a_c = 0.9$.

difference in the terminal velocities that drives the momentum and energy transfers decreases, and, the velocity variance decreases. In this figure, we have only shown the variance for the larger species. The variance for the smaller species can be calculated using (2.17), i.e. $m_1 \xi_1 = m_2 \xi_2$, and therefore follows similar trends. As the ratio of particle radii, a_2/a_1, approaches 1, the difference in the terminal velocities, $v_{1t}^{\dagger} - v_{2t}^{\dagger}$, is small compared to $v_{1t}^{\dagger}$. Since the ratio of timescales, υ, is proportional to $\upsilon_1(v_{1t}^{\dagger}/(v_{1t}^{\dagger} - v_{2t}^{\dagger}))^{\frac{2}{3}}$, υ is large even at small υ_1 and the asymptotic analysis is not valid. As a result, the curves have not been extended to $a_2/a_1 = 1$.

Figure 3 is a plot of the velocity variance ξ_1, non-dimensionalized by $(v_{1t}^{\dagger 2})$, as a function of the ratio of number densities n_2/n_1, with υ_1 fixed at 0.001. We would intuitively expect the velocity variance to increase with increasing n_2, due to an increase in the tranfer of momentum and energy per particle of species 1. However, figure 3 shows that the velocity variance first increases and then decreases. This is because the ratio of the leading-order velocity variances is inversely proportional to the ratio of the masses of the particles of the two species, and does not depend on the ratio of the number densities of the particles (see (2.17)). Thus, the velocity variances of the two species are coupled, and a decrease in the velocity variance of species 2, due to an increase in its number density, results in a decrease in the velocity variance of species 1 as well.

In figure 4 the $O(\upsilon)$ correct velocity variance, ξ_1, and the $O(\upsilon^2)$ correct radial and axial velocity variances, ξ_{1r} and ξ_{1z}, are plotted as functions of the number density ratio n_2/n_1. The size ratio is 0.7 and υ_1 is 0.001. The $O(\upsilon^2)$ vertical velocity variance, ξ_{1z}, shows a significant deviation from the leading-order variance, ξ_1, indicating that the $O(\upsilon^2)$ correction to the velocity variance is not small for υ greater than about 0.001. The large magnitude of the $O(\upsilon^2)$ correction suggests that the asymptotic expansion is only accurate for very small values of υ. The $O(\upsilon^2)$ corrections to the velocity variances are about 2–5% at $\upsilon_1 = 10^{-4}$, and the asymptotic analysis gives accurate results for υ of this order of magnitude.

The variances in figures 3 and 4 have not been extended to $n_2/n_1 = 0$ because the collision time, τ_{c12}, between collisions of a particle of species 1 with one of species 2,

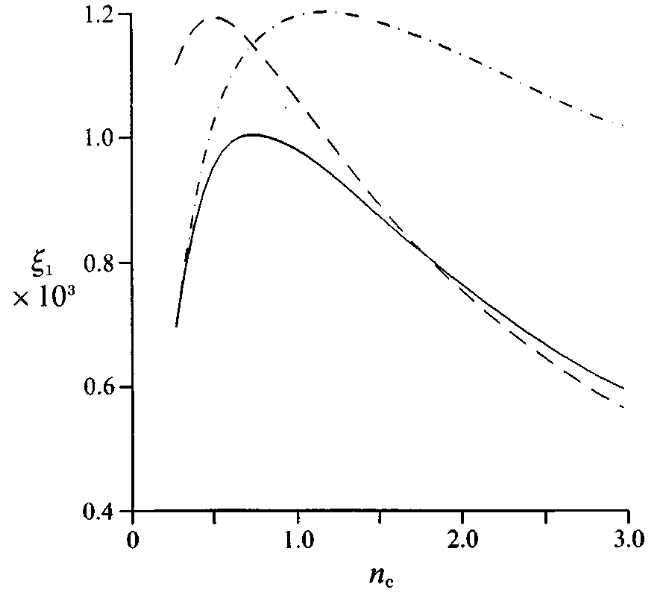

FIGURE 4. The isotropic velocity variance, ξ_1, correct to $O(\upsilon)$, and the velocity variances, ξ_{1r} and ξ_{1z}, correct to $O(\upsilon^2)$, non-dimensionalized by $v_{1t}^{\dagger 2}$, as a function of the number density ratio $n_c = n_2/n_1$. The ratio of particle radii a_2/a_1 is 0.7, and υ_1 is 0.001. ————, ξ_1; ——, ξ_{1r}; - - - - - - -, ξ_{1z}.

increases as the number of particles of species 2 is decreased, and the condition $\tau_{cij} \ll \tau_{vi}$ is no longer valid.

3. Suspension of inelastic particles

Inelastic collisions provide an additional mechanism of energy dissipation. In this section, we study the properties of the suspension in the limit where the coefficient of elasticity, e, is close to 1, but the dissipation of energy due to inelastic collisions is large compared to that due to viscous drag. In this limit, the direct effect of the inelasticity on the collision dynamics is small, and the elastic–particle collision integral is zero to leading order. The leading-order distribution functions are Gaussian distributions (2.16) whose variances, ξ_1 and ξ_2, are related by (2.17). Since momentum is conserved in inelastic collisions, the momentum balance equations for the present case are the same as those for a suspension of elastic particles. We can use the same scaling arguments as those used in §2.1 to show that the difference in the mean velocities of the two phases is $O(\upsilon)$ smaller than the mean velocity of the suspension. Equation (2.6) gives the leading-order mean velocity of the suspension. The collisional rate of change of mean velocity is given by

$$\frac{\partial_c \langle c_{iz} \rangle}{\partial t} = -\upsilon^{-\frac{1}{2}} \left(\frac{\tau_{c12}}{\tau_{cik}}\right) \left(\frac{e+1}{2}\right) \frac{8(\xi_1+\xi_2)^{\frac{1}{2}}}{3\pi^{\frac{1}{2}}} \frac{m_k}{m_i+m_k} (v'_{im} - v'_{km}). \tag{3.1}$$

The order of magnitude of the fluctuating velocities differs from that for a suspension of elastic particles, since there is a new dissipation mechanism. If the fluctuating velocities of the particles are $O(v_f^\dagger)$, the rate of dissipation of energy due to inelastic collisions is $O(\delta m_i v_f^{\dagger 2}/\tau_{cij})$, where δ is $1-e^2$. The rate of energy dissipation due to viscous drag is $O(m_i v_f^{\dagger 2}/\tau_{vi})$. Therefore, the dissipation rate due to inelastic collisions is $O(\delta/\upsilon)$ larger than the viscous dissipation rate in the limit $\delta \gg \upsilon$, and the analysis in the present section is valid in this limit. In the energy balance equations (2.9), the collisional source of energy scales as $m_i v_m^\dagger(v_{1m}^\dagger - v_{2m}^\dagger)/\tau_{vi}$, which is $O(\upsilon m_i v_m^{\dagger 2}/\tau_{vi})$ since the difference in the mean velocities is $O(\upsilon v_m^\dagger)$. It can be easily

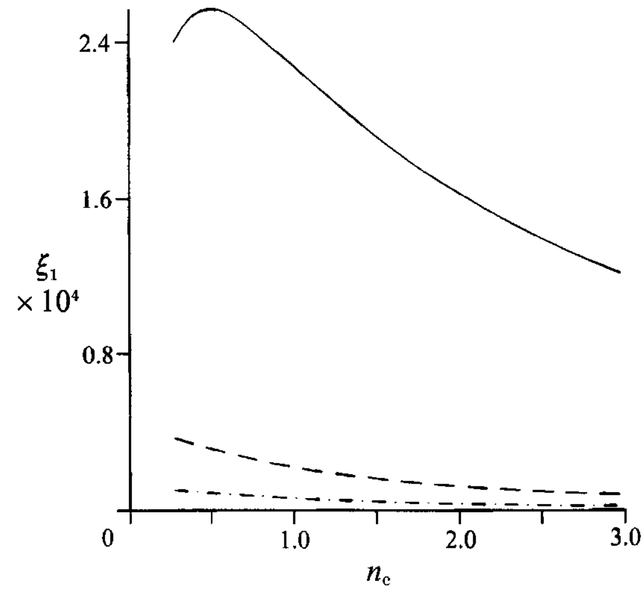

FIGURE 5. The velocity variance of particles of species 1, ξ_1, non-dimensionalized by $v_{1t}^{\dagger 2}$, as a function of ratio of number densities $n_c = n_2/n_1$, for a suspension of inelastic particles. The ratio of particle radii is 0.7, and the parameter υ_1, defined as $[(1-e^2)^{\frac{1}{4}}\,(n_1(4\pi a_1^2)\,v_{1t}^{\dagger}\tau_{v1})^{-\frac{1}{2}}]$, is 0.0001. The variance for $e = 1$ was calculated by setting $\upsilon_1 = 0.0001$ in the solution for elastic particles in §2. ——, $e = 1$; – – – –, $e = 0.95$; ·–·–·–·, $e = 0.85$.

verified that to achieve a balance between the collisional source and the inelastic dissipation, the fluctuating velocity should be $O(\upsilon/\delta^{\frac{1}{2}})$ smaller than the mean velocity. As in §2, υ is defined as the ratio of the collision time to the viscous relaxation time. However, the decrease in the magnitude of the velocity fluctuations due to inelastic collisions changes the dependence of υ on the radius and number density of the particles, and (2.5) is replaced by

$$\upsilon = (\tau_{c12}/\tau_{v1}) = \delta^{\frac{1}{4}}[n_2\,\pi d_{12}^2\,v_m^{\dagger}\,\tau_{v1}]^{-\frac{1}{2}}. \tag{3.2}$$

It can be easily seen that υ is proportional to $(\delta^{\frac{1}{4}}(St\,V)^{-\frac{1}{2}})$. Thus, there is an $O(\delta^{-\frac{1}{4}}\,(St\,V)^{-\frac{1}{6}})$ decrease in magnitude of the fluctuating velocity, $v_f^{\dagger}$, and an $O(\delta^{\frac{1}{4}}(St\,V)^{\frac{1}{6}})$ increase in the magnitude of the difference in the mean velocities of the two species, $v_{1m}^{\dagger}-v_{2m}^{\dagger}$, due to inelastic collisions. Note that the condition $\delta \gg \upsilon$ is equivalent to $\delta \gg (St\,V)^{-\frac{2}{3}}$.

In the balance equations for the mean-square fluctuating velocities (2.21), the energy dissipation due to inelastic collisions is large compared to that due to viscous drag. The total energy balance condition for an inelastic suspension is, instead of (2.24),

$$n_1 m_1 \frac{\partial_c \langle c_1^2\rangle}{\partial t} + n_2 m_2 \frac{\partial_c \langle c_2^2\rangle}{\partial t}$$
$$= -\delta\upsilon\left[2n_1 \frac{m_1 m_2}{m_1+m_2}\frac{(\xi_1+\xi_2)^{\frac{3}{2}}}{\pi^{\frac{1}{2}}} + \frac{n_1 m_1}{2}\left(\frac{\tau_{c12}}{\tau_{c11}}\right)\frac{(2\xi_1)^{\frac{3}{2}}}{\pi^{\frac{1}{2}}} + \frac{n_2 m_2}{2}\left(\frac{\tau_{c12}}{\tau_{c22}}\right)\frac{(2\xi_2)^{\frac{3}{2}}}{\pi^{\frac{1}{2}}}\right]. \tag{3.3}$$

The first term on the right-hand side of (3.3) is the dissipation of energy due to inelastic collisions between particles of species 1 and 2, and the last two terms are due to collisions between two particles of species 1 and two particles of species 2, respectively. The energy dissipation is calculated using the ensemble averaging technique derived in §2.4. The momentum balance equations for the two species, (2.21*a*), and the sum of the energy balance equations, (2.21*b*), for the two species, are solved to calculate the leading-order variance ξ_i, and the difference in mean velocities between the two species. Figure 5 shows the variation in the velocity variance, scaled by the terminal velocity of species 1, with the number density for three values of the coefficient of restitution.

4. Conclusions

The flow of a bidisperse suspension of solids, settling under gravity at steady state, was analysed in the limit where the time between successive collisions of a particle, τ_c, is small compared to its viscous relaxation time, τ_v. A small parameter, υ, was defined as the ratio of the two timescales τ_c/τ_v and a perturbation analysis was used in the limit of small υ. The scalings of the velocities, calculated from the momentum and energy balance equations, are as follows. The difference in the mean velocities of the two species, $v_{1m}^{\dagger} - v_{2m}^{\dagger}$, is $O(\upsilon v_m^{\dagger})$ where $v_m^{\dagger}$ is the mean velocity of the suspension, and the fluctuating velocity, $v_f^{\dagger}$, is $O(\upsilon^{\frac{1}{2}} v_m^{\dagger})$. Thus, the fluctuating velocity is $O(\upsilon^{\frac{1}{2}})$ smaller than the mean velocity of the suspension, and the difference in the mean velocities is $O(\upsilon^{\frac{1}{2}})$ smaller than the fluctuating velocity. Here υ is proportional to $(St\, V)^{-\frac{2}{3}}$, and the fluctuating velocity decreases as the volume fraction of the particles increases in this limit.

In the limit $\tau_c \ll \tau_v$, the accumulation of particles in velocity space due to collisions is $O(\upsilon^{\frac{1}{2}})$ larger than that due to the viscous drag forces. Therefore, the leading-order collisional accumulation is zero at steady state, and the leading-order distribution function is a Gaussian distribution, which is identical to the distribution of molecular velocities in a two-component gas at equilibrium. However, the energy required to sustain the fluctuations is provided by the drag force on the particles due to the difference between the mean and terminal velocities, and to calculate the velocity variances it is necessary to take into account the small effects of the viscous drag.

Balance equations for the velocity moments were derived using an anisotropic Gaussian distribution, which is a small perturbation to the leading-order distribution. Analytical expressions for the collisional transfer of momentum and energy between the two species, and the redistribution of energy between the horizontal and vertical velocity fluctuations, were derived using an ensemble averaging technique developed in §2.4. There is a transfer of momentum from the species with a higher mean velocity to that with a lower mean velocity, thus tending to equalize the mean velocities of the two species. The driving force for a collisional transfer of energy is the difference in the average energy per particle, $m_i \langle c_i^2 \rangle$, and this transfer is directed from the species having higher energy to that having lower energy per particle. There is also a transfer of energy between the horizontal and vertical directions, due to collisions between particles of the same species, which tends to equalize the average energies in the three coordinate directions.

The velocity variances were calculated correct to $O(\upsilon^2)$ from the higher-order balance equations for the velocity moments. For υ of $O(0.001)$, the $O(\upsilon^2)$ correction to the vertical velocity variance was found to be comparable to the $O(\upsilon)$ isotropic velocity variance, indicating that the perturbation series does not converge for this value of υ. The higher corrections to the velocity variances are small for υ less than about 10^{-4}, and the asymptotic analysis is valid for υ of this order of magnitude. In Kumaran & Koch (1993*b*) we calculate the velocity moments for higher values of υ using an approximate distribution.

The magnitude of the fluctuating velocity in a suspension of inelastic particles is small compared to that in an equivalent suspension of elastic particles in the limit

$(1-e^2) \gg \upsilon$, because the dissipation of energy due to inelastic collisions is large compared to that due to viscous drag. In this case, υ is proportional to $[(1-e^2)^{-\frac{1}{2}}(St\,V)^{-\frac{1}{2}}]$, and the limit $1-e^2 \gg \upsilon$ is equivalent to $1-e^2 \gg (St\;V)^{-\frac{2}{3}}$. The leading-order distribution functions of the two species are of the same form as those for elastic suspensions. However, the magnitude of the fluctuating velocity is $O[(1-e^2)^{-\frac{1}{2}}(St\;V)^{-\frac{1}{6}}]$ smaller than that in an elastic suspension, and the difference in the mean velocities of the two species is $O[(1-e^2)^{\frac{1}{2}}\,(St\,V)^{\frac{1}{6}}]$ larger than that in an elastic suspension. Thus, inelastic collisions could significantly decrease the magnitude of the fluctuating velocity even if the coefficient of restitution is close to 1.

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Appendix. Velocity distribution function for systems with velocity-dependent forces

Derivations of the steady-state velocity distribution function for gas molecules invoke the principle of detailed balancing, which states that, at the molecular level, the effect of any collisional interaction is exactly balanced by an inverse interaction so that the net effect is zero. The principle of detailed balancing is used to calculate the Maxwell–Boltzmann distribution of molecular velocities in a gas at equilibrium, and the corrections to the distribution function in a non-equilibrium gas. In the case of inelastic systems, it is known that the principle of detailed balancing breaks down because energy is not a collisional invariant. We show in this Appendix that this principle is not valid for suspensions of elastic particles at steady state when the divergence of the forces in velocity space is non-zero. Thus, the distribution function for a bidisperse particle–gas suspension at steady state could be very different from the Maxwell–Boltzmann distribution when the effect of the drag forces is significant, and we cannot use the type of perturbation expansions that are developed for small deviations from equilibrium in the kinetic theory of gases.

The conservation equation for the molecular distribution function is the Boltzmann equation, which is derived in chapter 3 of Chapman & Cowling (1970). In their chapter 4, the Boltzmann H-theorem is invoked to show that the principle of detailed balancing is valid for the system at steady state. This principle is used to derive the Maxwell–Boltzmann distribution. The derivations in this Appendix run parallel to those in chapters 3 and 4 of Chapman & Cowling, but here we consider a two-component system.

Owing to the presence of velocity-dependent forces, we modify the Boltzmann equation (3.1) for particles of species i by including the forces within the divergence operator:

$$\frac{\partial f_i}{\partial t}+\boldsymbol{c}_{\mathrm{i}}\cdot\frac{\partial f_i}{\partial \boldsymbol{r}_i}+\frac{\partial}{\partial \boldsymbol{c}_i}\cdot(\boldsymbol{F}_i f_i)=\frac{\partial_{\mathrm{c}} f_i}{\partial t}. \tag{A 1}$$

Here the acceleration $\boldsymbol{F}_i$ is a function of the fluctuating particle velocity $\boldsymbol{c}_i$, and $\partial_{\mathrm{c}} f_i/\partial t$ is the collision integral, which is the rate of change of the distribution function due to collisions between particles. The derivation proceeds by defining the quantity H as follows:

$$\frac{\mathrm{d}H}{\mathrm{d}t}=\sum_i\int(1+\log f_i)\frac{\partial f_i}{\partial t}\mathrm{d}\boldsymbol{c}_i. \tag{A 2}$$

The summation in (A 2) is over the species 1 and 2. The rate of change of H, equivalent to (4.1.1) of Chapman & Cowling, is given by

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum_i \int (1+\log f_i)\left[\frac{\partial_c f_i}{\partial t} - \boldsymbol{c}_i \cdot \frac{\partial f_i}{\partial \boldsymbol{r}_i} - \frac{\partial}{\partial \boldsymbol{c}_i}\cdot(\boldsymbol{F}_i f_i)\right]\mathrm{d}\boldsymbol{c}_i$$

$$= \sum_i \int \left[\frac{\partial_c (f_i \log f_i)}{\partial t} - \boldsymbol{c}_i \cdot \frac{\partial (f_i \log f_i)}{\partial \boldsymbol{r}_i} - \frac{\partial}{\partial \boldsymbol{c}_i}\cdot(\boldsymbol{F}_i f_i \log f_i) - f\left(\frac{\partial}{\partial \boldsymbol{c}_i}\cdot \boldsymbol{F}_i\right)\right]\mathrm{d}\boldsymbol{c}_i. \quad (\mathrm{A}\ 3)$$

As shown, in §§4.13 and 4.14 of Chapman & Cowling, the second term in the above integral vanishes for spatially uniform suspensions or for suspensions in vessels with smooth walls, and the third term vanishes if the distribution function goes to zero exponentially at large velocities. The last term, however, does not sum to zero in the most general case, and (4.1.4) of Chapman & Cowling has to be modified to take this term into account:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{1}{4}\left[\sum_i \sum_j \int \log\left(\frac{f_i f_j}{f'_i f'_j}\right)(f'_i f'_j - f_i f_j)\, k_j \,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{c}_i\,\mathrm{d}\boldsymbol{c}_j\right] - \sum_i \int f_i \left(\frac{\partial}{\partial \boldsymbol{c}_i}\cdot \boldsymbol{F}_i\right)\mathrm{d}\boldsymbol{c}_i. \quad (\mathrm{A}\ 4)$$

The prime is used to indicate the distribution function after collision, and $k_j\,\mathrm{d}\boldsymbol{k}$ is the differential collision cross-section. At steady state, the rate of change of H is zero, and the above equation reduces to

$$\sum_i \sum_j \int \log\left(\frac{f_i f_j}{f'_i f'_j}\right)(f'_i f'_j - f_i f_j)\, k_j \,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{c}_i\,\mathrm{d}\boldsymbol{c}_j = 4\sum_i \int f_i \left(\frac{\partial}{\partial \boldsymbol{c}_i}\cdot \boldsymbol{F}_i\right)\mathrm{d}\boldsymbol{c}_i. \quad (\mathrm{A}\ 5)$$

The left-hand side of (A 5) is always negative, because $\log(f_i f_j / f'_i f'_j)$ is always opposite in sign to $f'_i f'_j - f_i f_j$. The right-hand side of (A 5) is negative for viscous drag forces, and there exists a possible steady-state solution for the distribution function that is not a Gaussian. If the force is in the same direction as the particle velocity, however, the right-hand side of (A 5) is positive and the left-hand side is negative, indicating that there is no possible steady-state solution for the distribution function. This agrees with what we would expect intuitively, since a force that acts in the direction of velocity increases the energy of the system indefinitely.

For certain systems, such as the flow of charged particles in a magnetic field, the divergence of the force in velocity space is zero even though the forces are velocity dependent. The magnetic force on the particles is $\boldsymbol{F}_i + (e/m)\,\boldsymbol{c}_i \wedge \boldsymbol{H}$ where $\boldsymbol{F}_i$ is a constant force, $\boldsymbol{H}$ is the magnetic field intensity, e and m are the charge and mass of an electron respectively and $\wedge$ is the vector cross-product. The divergence of this force in velocity space is zero, and, therefore, the principle of detailed balancing is valid, as indicated in chapter 18 of Chapman & Cowling.

The breakdown of the principle of detailed balancing can be explained as follows. Particle accumulation is caused by the acceleration of particles by external forces, and by the instantaneous change in the particle velocity due to collisions. If the external forces on the particles are independent of particle velocity, there is no net accumulation of particles in a differential volume $\mathrm{d}\boldsymbol{c}_i$ due to the forces. Therefore, at steady state, the number of particles entering the differential volume $\mathrm{d}\boldsymbol{c}_i$ and the number of particles leaving it due to collisions are equal, i.e. the collisions as a whole produce no net effect on the distribution function. Under this condition, the Boltzmann H-Theorem can be used to show that for every collision in which the colliding particles have initial velocities $\boldsymbol{c}_i$ and $\boldsymbol{c}_j$ and final velocities $\boldsymbol{c}'_i$ and $\boldsymbol{c}'_j$, there is an inverse collision between particles with initial velocities $\boldsymbol{c}'_i$ and $\boldsymbol{c}'_j$ and final velocities $\boldsymbol{c}_i$ and $\boldsymbol{c}_j$, i.e. the effect of every collision is exactly balanced by the effect of an inverse collision. If the external force is dependent on particle velocity, however, there is a net accumulation of particles due to the force. Therefore, the collisional scattering into and out of this differential volume are not equal at steady state. Since the collisional effects are not balanced in a global sense, the principle of detailed balancing is not valid for individual collisions between particles.

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