J. Fluid Mech. (1996), *vol.* 320, *pp.* 1–17

Stability of inviscid flow in a flexible tube

By V. KUMARAN

Department of Chemical Engineering, Indian Institute of Science, Bangalore 560 012, India

(Received 22 May 1995 and in revised form 9 November 1995)

The equivalents of the classical theorems of hydrodynamic stability are derived for inviscid flow through a flexible tube. An important difference between flows in plane and cylindrical geometries is that the Squire transformation, which states that two-dimensional perturbations in plane parallel flows are always more unstable than three-dimensional perturbations, is not valid for tube flows. Therefore, it is necessary to analyse both axisymmetric and non-axisymmetric perturbations in flows in cylindrical geometries. Perturbations of the form $v_i = \tilde{v}_i \exp[ik(x-ct)+in\phi]$ are imposed on a steady axisymmetric mean flow $V(r)$, and the stability of the mean velocity profiles and bounds for the phase velocity of the unstable modes are determined. Here r, ϕ and x are the radial, polar and axial directions, and k and c are the wavenumber and phase velocity. The flexible wall is represented by a standard constitutive equation which contains inertial, elastic and dissipative terms. Results for general velocity profiles are derived in two limiting cases: axisymmetric flows ($n = 0$) and highly non-axisymmetric flows ($n \gg k$). The results indicate that axisymmetric perturbations are always stable for $(V''-r^{-1}V')V \geqslant 0$ and could be unstable for $(V''-r^{-1}V')V < 0$, while highly non-axisymmetric perturbations are always stable for $(V''+r^{-1}V')V \geqslant 0$ and could be unstable for $(V''+r^{-1}V')V < 0$. In addition, bounds on the real part (c_r) and imaginary part (c_i) of the phase velocity are also derived. For the practically important case of Hagen–Poiseuille flow, the present analysis indicates that axisymmetric perturbations are always stable, while highly non-axisymmetric perturbations could be unstable. This is in contrast to plane parallel flows where two-dimensional disturbances are always more unstable than three-dimensional ones.

1. Introduction

The flow of a fluid in a flexible tube is encountered in many biological systems and biotechnology applications. Accurate predictions for the heat and mass transfer rates in these systems could be of use in optimizing process design. The transfer rates depend on the flow regime, and the transport coefficients in a turbulent flow are typically three orders of magnitude greater than those in a laminar flow. Thus, a good understanding of the transition from laminar to turbulent flow could be of importance in practical applications. In processes where high heat or mass transfer rates are desired, the process could be operated in the turbulent regime, while in systems where a low drag force is of importance, the laminar flow would be more suitable. The analysis of the hydrodynamic stability of the basic laminar flow is of interest because the laminar to turbulent transition is usually preceded by an instability of the laminar flow. In this study, the stability of inviscid flow in a flexible tube to both axisymmetric and non-axisymmetric perturbations is studied, and the equivalents of some of the classical results for parallel shear flows for general velocity profiles are derived.

The stability of the flow through a two-dimensional rigid-walled channel is constrained by the Rayleigh theorem, which states that an inflection point is necessary for unstable modes to exist. In addition, there are results due to Rayleigh, Howard and Høiland (Drazin & Reid 1981) which place bounds on the real and imaginary parts of the phase velocity of unstable modes. Most two-dimensional flows encountered in practice, such as plane Poiseuille flow in a channel and boundary layer flows, do not have inflection points, and the Rayleigh theorem predicts that these flows are always stable. However, there are certain solutions of the complete Navier–Stokes equations which do not reduce to the solutions of the inviscid equations. These solutions, which involve an inner critical layer, where viscous forces are important, could become unstable when the Reynolds number is increased beyond a critical value. The fluid viscosity acts as a destabilizing effect for these modes, and this type of instability is called the Tollmien–Schlichting instability. In contrast, there is no critical layer in axisymmetric Hagen–Poiseuille flow in a tube, and so there is no destabilization due to viscous effects. The asymptotic analyses of Corcos & Sellars (1959) and Gill (1965) of high Reynolds number flow in a rigid tube showed that the flow is always stable. In addition, there have been many numerical studies of the linear stability of axisymmetric and non-axisymmetric perturbations for Hagen–Poiseuille flow in a rigid tube (Davey & Drazin 1969; Garg & Rouleau 1972; Salwen & Grosch 1972) which have concluded that the flow is stable to small disturbances at all Reynolds numbers. The experimentally observed instability at a Reynolds number of about 2300 (see, for example, Wygnanski & Champagne 1973) is considered to be due to finite amplitude perturbations, and if sufficient precautions are taken to prevent fluctuations in the apparatus, the flow can be maintained in the laminar regime at much higher Reynolds numbers.

The study of the stability of flows over compliant walls has been motivated by the desirability of drag reduction in marine and aerospace applications. Following the experiments of Kramer (1960), Benjamin (1959, 1963) and Landahl (1962) used an extension of the theory of Tollmien (1929) to study the stability of the flow past a compliant surface. Benjamin found that the presence of a flexible wall tends to stabilize the Tollmien–Schlichting instability, but dissipation in the wall has a destabilizing effect. In addition, there is a type of instability called the flow-induced surface instability which is not present in the flow past a rigid surface. The earlier studies of Benjamin and Landahl used simple models such as those for membranes and plates. More complicated wall models which involve a plate supported on an isotropic or anisotropic elastic foundation backed by a liquid substrate have been studied Carpenter and coworkers (see, for example, Carpenter & Garrad 1985, 1986; Carpenter & Gajjar 1990; and the review article by Carpenter 1990).

There is also experimental evidence to indicate that the stability of the flow in a flexible tube is affected by the wall dynamics. Krindel & Silberberg (1979) studied the flow of a fluid through a gel-walled tube, and reported that the Reynolds number for the transition from laminar to turbulent flow could be much lower than the value of 2300 for a rigid tube. Evrensel *et al.* (1993) carried out numerical studies of the flow of air in a tube with a compliant wall to simulate the experiments of King, Brock & Lundell (1985) on the flow in pulmonary airways. They found that the transition Reynolds number was about 150 in an elastic tube, but increased significantly to about 550 when viscous dissipation was included in the equations for the wall. In both the studies, the transition Reynolds number was found to depend on the elasticity of the wall in addition to the fluid properties, indicating that the wall dynamics has a significant influence on the flow. The experiments of Krindel & Silberberg also found that the variation in the drag force at the transition is gradual, in contrast to the neardiscontinuous change in the drag in a rigid tube. This suggests that the destabilizing mechanism in a flexible tube could be different from that in a rigid tube.

The stability of viscous flow in a flexible tube was analysed by the author (Kumaran 1995*a*) using a linear stability analysis. It was found that even in the absence of inertial effects, the flow could become unstable when the Reynolds number was increased beyond a critical value. The instability is caused by an additional term in the tangential velocity boundary condition at the interface between the fluid and the wall, which represents the variation in the mean velocity at the surface due to the surface displacement. The destabilizing mechanism is the transport of energy from the mean flow to the fluctuations due to the shear work done by the mean flow at the interface. The stability of axisymmetric perturbations in high Reynolds number flow in a flexible tube was also studied by the author (Kumaran 1995*b*). In this regime, the flow in the core of the tube is inviscid, but viscous effects are important in a boundary layer of thickness $Re^{-1/2}$ at the walls. An asymptotic analysis in the small parameter Re^{-1} was used. In the leading approximation, the perturbations were found to be neutrally stable because there is no dissipation of energy in the inviscid flow, and no transport of energy from the mean flow to the fluctuations. The first correction to the growth rate due to viscous effects was also calculated, and it was found that the real part of the first correction to the growth rate is always negative, indicating that the flow is stable.

The high Reynolds number analysis is not conclusive, however, since only axisymmetric disturbances were considered. In addition, there are regions near the entrance of the tube where the flow profile is different from fully developed Hagen–Poiseuille flow, and the mean velocity in tubes of slowly varying cross-section could also be very different from Hagen–Poiseuille flow. Therefore, it is of interest to consider more general velocity profiles to draw definite conclusions about the stability limits. The computational effort required for determining the stability of non-axisymmetric modes is likely to be quite considerable owing to the presence of an additional parameter, and there do not appear to have been any systematic computational studies until now. The work involved would be greatly reduced if general criteria could be derived for potentially unstable velocity profiles, and bounds could be obtained for the growth rate and wave speed of the unstable modes. There are results due to Rayleigh, Fjørtoft, Howard and Høiland (Drazin & Reid 1981) which predict potentially unstable velocity profiles and provide bounds on the phase velocity of the unstable mode for two-dimensional flows bounded by rigid walls. In the present analysis, these results are extended to flow in a flexible tube.

The stability of inviscid flow is sensitively dependent on the boundary conditions at the surface, and the stability of flow near a rigid surface is very different from that near a flexible surface. The Rayleigh inflection point theorem, which states that unstable modes can exist only if the mean flow has an inflection point, is not applicable in a flexible channel because a non-zero normal velocity is permitted at the wall. The stability of inviscid flows in channels with passive compliant walls was analysed by Yeo & Dowling (1987) and Yeo (1994). A general constitutive equation for a compliant wall was derived using a variational-Lagrangian formulation, and this was used to relate the fluid velocity and stress at the wall. Bounds on the real and imaginary parts of the phase velocity were derived, and it was shown that the experimental observations of Gad-el-Hak, Blackwelder & Riley (1984) and Hansen *et al.* (1980) conformed to these bounds. In the present study, we examine the extent to which the classical results of hydrodynamic stability, and the results of Yeo & Dowling (1987) and Yeo (1994), can be extended to the flow through a flexible tube.

There is one additional complication in tube flow which is not present in two-dimensional flow. In the latter, the Squire transformation (Drazin & Reid 1981) can be invoked to show that a two-dimensional perturbation is always more unstable than a three-dimensional perturbation. The Squire transformation cannot be extended to a general two-dimensional flow near a flexible surface, but Rotenberry & Saffman (1990) have shown that two-dimensional disturbances are more unstable than three-dimensional disturbances for the specific case of channel flow when the stiffness, rigidity and damping coefficient of the wall are independent of the wavenumber of the perturbation. As a consequence, it is sufficient to analyse two-dimensional disturbances to derive stability criteria for these situations. There is no equivalent of the Squire transformation for tube flow, and non-axisymmetric modes could be more unstable than axisymmetric modes. It is possible to derive general results in two limiting cases. Axisymmetric modes are analysed in the next section, and 'highly non-axisymmetric' modes, where the gradients in the polar direction are large compared to those in the axial direction, are considered in §3. Results for the stability of parallel flows are usually derived using the stream function formulation because only two-dimensional disturbances are considered, and the fourth-order Orr–Sommerfeld equation for the stream function is used as the starting point. In the present case, it is more convenient to express the equations in terms of the velocity itself because axisymmetric and non-axisymmetric perturbations are considered. Despite this, the derivations are similar to those used for the classical results of hydrodynamic stability.

2. Axisymmetric perturbations

Consider an incompressible Newtonian fluid flowing through a tube of radius R bounded by a flexible surface as shown in figure 1. The mean velocity $V(r)$ is axisymmetric and steady, and decreases to zero at the wall. A small-amplitude axisymmetric normal mode perturbation for the velocity has components v_r and v_x in the radial and axial directions of the form

$$v_i = \tilde{v}_i(r)\exp[\mathrm{i}k(x-ct)], \tag{2.1}$$

where k is the real wavenumber of the perturbations, and c is the complex wave speed and we assume $k \geqslant 0$ without loss of generality. The imaginary part of the wave speed, c_i, is positive for unstable perturbations and negative for stable perturbations. The mass and momentum equations for an axisymmetric flow are

$$(\mathrm{d}_r + r^{-1})\tilde{v}_r + \mathrm{i}k\tilde{v}_x = 0, \tag{2.2}$$

$$\mathrm{d}_r\tilde{p} + \mathrm{i}k(V-c)\tilde{v}_r = 0, \tag{2.3}$$

$$\mathrm{i}k\tilde{p} + \mathrm{i}k(V-c)\tilde{v}_x + V'\tilde{v}_r = 0, \tag{2.4}$$

where $\mathrm{d}_r \equiv (\mathrm{d}/\mathrm{d}r)$, $\tilde{p}$ is the pressure and $V' = \mathrm{d}_r V$. The fluid density ρ does not explicitly appear in the above equation because the pressure has been scaled by the density. In the remainder of the analysis, all quantities are scaled by an appropriate combination of the density ρ, a velocity scale of the mean flow V_m and the radius of the tube R, so all equations are dimensionless. The above equations can be combined by adding $(\mathrm{i}k \times (2.3) - \mathrm{d}_r \times (2.4))$, and using (2.2) to express $\tilde{v}_x$ in terms of $\tilde{v}_r$, to obtain the second-order Rayleigh-type equation for $\tilde{v}_r$:

$$(\mathrm{d}_r^2 + r^{-1}\mathrm{d}_r - r^{-2} - k^2)\tilde{v}_r - \frac{V'' - r^{-1}V'}{V-c}\tilde{v}_r = 0. \tag{2.5}$$

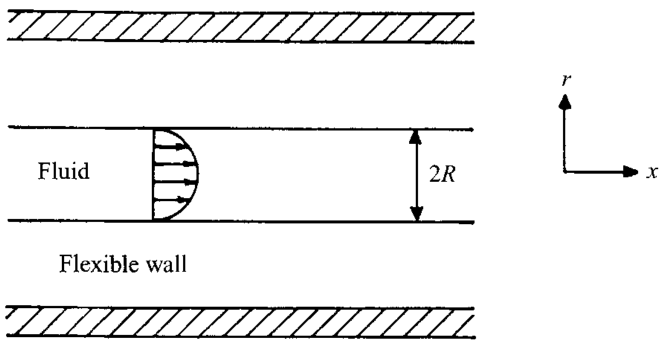

FIGURE 1. Configuration and definition of the coordinate system.

Equation (2.5) is now multiplied by $r\tilde{v}_r^*$, where * denotes the complex conjugate, and integrated from 0 to 1 to obtain

$$\int_0^1 r\,\mathrm{d}r\left[|\mathrm{d}_r\tilde{v}_r|^2+\left(k^2+r^{-2}+\frac{(V''-r^{-1}V')(V-c^*)}{|V-c|^2}\right)|\tilde{v}_r|^2\right] = (r\tilde{v}_r^*\,\mathrm{d}_r\tilde{v}_r)|_{r=1}-(r\tilde{v}_r^*\,\mathrm{d}_r\tilde{v}_r)|_{r=0}. \quad (2.6)$$

The second term on the right-hand side of (2.6) is zero because the radial velocity is zero at the centre of the tube.

The dynamics of the wall is represented by the normal displacement field, u, which is the displacement of the wall from its steady-state position due to the velocity fluctuations. In the linear analysis, the displacement field also has a normal mode form:

$$u = \tilde{u}\exp[\mathrm{i}k(x-ct)]. \quad (2.7)$$

For small displacements, the normal stress is a linear function of the displacement field. Here, the constitutive equation is considered to be of the form

$$\tilde{\sigma} = -(E-k^2c^2I-\mathrm{i}kcD)\,\tilde{u}, \quad (2.8)$$

where E, I and D are the positive constants associated with the elasticity, inertia and dissipation in the wall. These are in general functions of k, c and the wall properties. Equation (2.8) was derived by Yeo & Dowling (1987) for a passive compliant wall using a variational Lagrangian formulation of the wall dynamics. The right-hand side of (2.8) differs from that of Yeo & Dowling (1987) by a negative sign because $\tilde{u}$ is considered positive when directed outward along the radial direction.

The boundary conditions at the interface between the wall and fluid are the continuity of velocity and normal stress:

$$-\mathrm{i}kc\tilde{u} = \tilde{v}_r, \quad (2.9)$$

$$\tilde{\sigma} = -\tilde{p} = (V-c)\,\tilde{v}_x+\tilde{v}_r\,V'/(\mathrm{i}k), \quad (2.10)$$

where it has been assumed that $k \geqslant 0$ without loss of generality. Using the above relations, the right-hand side of (2.6) can be expressed as

$$(r\tilde{v}_r^*\,\mathrm{d}_r\tilde{v}_r)|_{r=1} = \left[-1-\frac{V'_w}{c}+\frac{E-k^2c^2I-\mathrm{i}kcD}{c^2}\right](\tilde{v}_r\tilde{v}_r^*)|_{r=1}, \quad (2.11)$$

where V'_w is the mean strain rate at the wall, and the mean velocity at the wall has been set equal to zero. Inserting (2.11) into (2.6), multiplying the resulting equation by c and taking the imaginary part, we get

$$c_i \int_0^1 r\,\mathrm{d}r \left[|\mathrm{d}_r \tilde{v}_r|^2 + |\tilde{v}_r|^2 \left(r^{-2} + k^2 + \frac{(V'' - r^{-1}V')\,V}{|V-c|^2} \right) \right] = -\left[c_i \left(1 + \frac{E}{|c|^2} + k^2 I \right) + kD \right] (\tilde{v}_r \tilde{v}_r^*)|_{r=1}. \quad (2.12)$$

For an unstable mode ($c_i > 0$), the right-hand side of (2.12) is negative because E, I and D are positive and $k \geqslant 0$. The left-hand side can be negative only if $(V'' - r^{-1}V')\,V$ is negative at some point in the flow. This leads to the equivalent of the Rayleigh inflection point theorem for axisymmetric perturbations in a flexible tube:

PROPOSITION 1. *For an inviscid flow in a flexible-walled tube with $V_w = 0$, an unstable axisymmetric mode can exist only if $(V'' - r^{-1}V')\,V < 0$ somewhere in the flow.*

The above result is also a modification of Proposition 4 of Yeo (1994) for the flow near a flexible surface. For the practically important case of Hagen–Poiseuille flow in a tube, $V'' - r^{-1}V'$ is identically zero, and the above proposition predicts that there are no unstable modes. In addition, two further conclusions can be drawn from (2.12):

COROLLARY 1. *For an inviscid flow in a flexible-walled tube with $V_w = 0$ and $(V'' - r^{-1}V')\,V \geqslant 0$, all axisymmetric modes are neutrally stable if the wall is non-dissipative ($D = 0$).*

COROLLARY 2. *For an inviscid flow in a flexible-walled tube with $V_w = 0$ and $(V'' - r^{-1}V')\,V \geqslant 0$, all axisymmetric modes are damped if the wall is dissipative ($D > 0$).*

Another result can be derived by inserting (2.11) into (2.6), and taking the imaginary part of the resulting equation:

$$c_i \int_0^1 r\,\mathrm{d}r \left[\frac{V'' - r^{-1}V'}{|V-c|^2} \tilde{v}_r \tilde{v}_r^* \right] = -\left[c_i \left(\frac{2Ec_r}{|c|^4} - \frac{V_w'}{|c|^2} \right) + \frac{kDc_r}{|c|^2} \right] (\tilde{v}_r \tilde{v}_r^*)|_{r=1}. \quad (2.13)$$

For a flow with $V_w' \leqslant 0$, the right-hand side of (2.13) is negative for unstable modes with positive phase speed $c_r > 0$. Such an instability can exist only if $(V'' - r^{-1}V') < 0$ somewhere in the flow. This gives the equivalent of the Fjørtoft theorem (and Proposition 5 of Yeo 1994):

PROPOSITION 2. *For axisymmetric flow in a flexible tube with $V_w = 0$ and $V_w' \leqslant 0$ ($V_w' \geqslant 0$), an unstable mode with $c_r \geqslant 0$ ($c_r \leqslant 0$) can exist only if $(V'' - r^{-1}V') < 0$ ($V'' - r^{-1}V > 0$) somewhere in the flow.*

Bounds for the real part of the wavenumber c_r of potentially unstable waves can be derived in a manner similar to Rayleigh's result (Drazin & Reid 1981) for the flow in a rigid-walled channel. Consider the function $f(r)$ defined by

$$f(r) = \frac{\tilde{v}_r}{V - c}. \quad (2.14)$$

The Rayleigh equation (2.5) can be written in terms of f as

$$\mathrm{d}_r[(V-c)^2 \mathrm{d}_r f] + r^{-1}(V-c)^2 \mathrm{d}_r f + 2r^{-1}V'(V-c)f - (r^{-2}+k^2)(V-c)^2 f = 0. \quad (2.15)$$

The above equation is multiplied by rf^* and integrated by parts:

$$\int_0^1 r\,\mathrm{d}r\,[(V-c)^2(|\mathrm{d}_r f|^2+(r^{-2}+k^2)|f|^2)-2r^{-1}V'(V-c)|f|^2] = [(V-c)^2 f^*\,\mathrm{d}_r f]_{r=1}. \tag{2.16}$$

The right-hand side of (2.16) can be simplified using (2.9) and (2.10):

$$[(V-c)^2 f^*\,\mathrm{d}_r f]_{r=1} = [E-k^2c^2 I-\mathrm{i}kcD-c^2](ff^*)|_{r=1}. \tag{2.17}$$

When (2.17) is inserted into the right-hand side of (2.16), the imaginary part of the resulting equation is

$$\int_0^1 r\,\mathrm{d}r\,[-2c_i(V-c_r)(|\mathrm{d}_r f|^2+(r^{-2}+k^2)|f|^2)+2r^{-1}c_i V'|f|^2]$$
$$= [-2c_r c_i(1+k^2 I)-kc_r D](ff^*)|_{r=1}. \tag{2.18}$$

From (2.18), certain conclusions can be drawn about the range of the real part of the wave speed, c_r, for flows where the velocity is zero at the wall and varies monotonically towards the centre of the tube. First, consider the case where $V \geqslant 0$ and $V' \leqslant 0$ in the tube. For unstable upstream-travelling waves with $c_r < 0$ and $c_i > 0$, the left-hand side of (2.18) is negative while the right-hand side is positive, and the equality is never satisfied. Therefore, all unstable modes must have $c_r > 0$. An upper bound on c_r can be obtained as follows. For unstable downstream-travelling waves with $c_r > \max(V)$ and $c_i > 0$, the right-hand side of (2.18) is negative since I and D are positive and $k \geqslant 0$. Using the inequality

$$\int_0^1 r\,\mathrm{d}r\,[(c_r-V)(|\mathrm{d}_r f|^2+(r^{-2}+k^2)|f|^2)+r^{-1}V'|f|^2]$$
$$> \int_0^1 r\,\mathrm{d}r\,[c_r(k^2+r^{-2})|f|^2-(V(k^2+r^{-2})-r^{-1}V')|f|^2], \tag{2.19}$$

a necessary condition for an unstable mode is that the integral on the right has to be negative. In order to obtain a bound on c_r, it is necessary to use the symmetry condition $\tilde{v}_r = 0$ and that $\mathrm{d}_r\tilde{v}_r$ is finite at the centre of the tube $r=0$. Therefore, the function f can be expressed as $f = rF$, where F is finite at the centre of the tube. When the term on the right in (2.19) is expressed in terms of F, the following inequality is a necessary condition for unstable modes:

$$\int_0^1 r\,\mathrm{d}r\,[c_r(k^2r^2+1)|F|^2-(V(k^2r^2+1)-rV')|F|^2] < 0. \tag{2.20}$$

For the above inequality to be valid, the integrand has to be negative at at least one point in the flow, and it is necessary that $c_r < \max[V-rV'/(k^2r^2+1)]$. This provides an upper bound on the wave speed c_r. A symmetrical argument can be used for the case $V \leqslant 0$ and $V' \geqslant 0$, and we have

PROPOSITION 3. *For an inviscid flow in a flexible-walled tube with $V_w = 0$ and $V' \leqslant 0$ ($V' \geqslant 0$) in the tube, all unstable axisymmetric modes have*

$$0 < c_r < \max\left[V-\frac{rV'}{k^2r^2+1}\right] \quad \left(\min\left[V-\frac{rV'}{k^2r^2+1}\right] < c_r < 0\right).$$

A bound on the magnitude of the phase velocity $|c|$ for axisymmetric perturbations can be obtained for a monotonically varying flow with $V_w = 0$ and $VV' < 0$. Substituting (2.17) for the right-hand side of (2.16), multiplying the resulting equation by c^* and taking the imaginary part, we get

$$\int_0^1 r\,\mathrm{d}r\,[c_i(|c|^2 - V^2)(|\mathrm{d}_r f|^2 + (r^{-2}+k^2)|f|^2) + 2r^{-1}c_i V' V |f|^2]$$
$$= -[c_i E + c_i(k^2 I + 1)|c|^2 + k|c|^2 D](ff^*)|_{r=1}. \quad (2.21)$$

For unstable waves with $c_i > 0$, the right-hand side of (2.21) is negative since E, I, D and k are positive. Using arguments similar to those leading to Proposition 3, it can easily be seen that

PROPOSITION 4. *For an inviscid flow in a flexible-walled tube with $V_w = 0$ and $VV' < 0$ in the tube, all unstable axisymmetric modes have*

$$|c|^2 < \max\left[V^2 - \frac{2rVV'}{k^2r^2+1}\right].$$

The theorem of Høiland (Drazin & Reid 1981) can be extended to axisymmetric perturbations using methods similar to that for the flow past a rigid wall. The function $g(r)$ is defined as

$$g(r) = \frac{\tilde{v}_r}{(V-c)^{1/2}}. \quad (2.22)$$

The conservation equation (2.5) expressed in terms of g is

$$\mathrm{d}_r[(V-c)\,\mathrm{d}_r g] + r^{-1}(V-c)\,\mathrm{d}_r g - (r^{-2}+k^2)(V-c)g - \frac{gV''}{2} - \frac{g(V')^2}{4(V-c)} + \frac{3r^{-1}V'g}{2} = 0. \quad (2.23)$$

Equation (2.23) is multiplied by rg^* and integrated from $r=0$ to $r=1$ to give

$$\int_0^1 r\,\mathrm{d}r\left[(V-c)(|\mathrm{d}_r g|^2 + (r^{-2}+k^2)|g|^2)\right.$$
$$\left. + \left(\frac{V''}{2} + \frac{(V')^2}{4(V-c)} - \frac{3r^{-1}V'}{2}\right)|g|^2\right] = (g^*(V-c)\,\mathrm{d}_r g)|_{r=1}. \quad (2.24)$$

The right side of (2.24) can be simplified using (2.9) and (2.10) to express $\mathrm{d}_r\tilde{v}_r$ in terms of $\tilde{v}_r$:

$$(g^*(V-c)\,\mathrm{d}_r g)|_{r=1} = \left[-\frac{E - c^2(1+k^2I) - \mathrm{i}kcD}{c} + \frac{V'}{2}\right](gg^*)|_{r=1}. \quad (2.25)$$

Inserting (2.25) into (2.24), taking the imaginary part of the resulting equation and multiplying by c_i, we obtain

$$\int_0^1 r\,\mathrm{d}r\left[-c_i^2(|\mathrm{d}_r g|^2 + (r^{-2}+k^2)|g|^2) + \frac{c_i^2(V')^2|g|^2}{4|V-c|^2}\right]$$
$$= \left[\frac{c_i^2 E + |c|^2 c_i^2(k^2I+1) + k|c|^2 c_i D}{|c|^2}\right](gg^*)|_{r=1}. \quad (2.26)$$

The right-hand side of (2.26) is positive for unstable waves with $c_i > 0$, and a necessary condition for the presence of unstable waves is that the left-hand side should also be positive. Using the inequality

$$\int_0^1 r\,\mathrm{d}r\left[-c_i^2(|\mathrm{d}_r g|^2+(r^{-2}+k^2)|g|^2)+\frac{c_i^2(V')^2|g|^2}{4|V-c|^2}\right] < \int_0^1 r\,\mathrm{d}r\left[-c_i^2(r^{-2}+k^2)|g|^2+\frac{c_i^2(V')^2|g|^2}{4|V-c|^2}\right], \quad (2.27)$$

a necessary condition for the presence of unstable modes is that the integral on the right should be positive. The symmetry conditions at the centre of the tube require that g can be expressed as $g = rG$, where G is finite at the centre of the tube. Using this substitution, the necessary condition for unstable modes becomes

$$\int_0^1 r\,\mathrm{d}r\left[-c_i^2(k^2r^2+1)|G|^2)+\frac{c_i^2r^2(V')^2|G|^2}{4|V-c|^2}\right] > 0. \quad (2.28)$$

For the above inequality to be valid, the integrand on the left-hand side has to be positive at at least one point in the flow, and the following inequality has to hold for the phase velocity:

$$c_i^2 < \max\left(\frac{r^2(V')^2c_i^2}{4|V-c|^2(k^2r^2+1)}\right). \quad (2.29)$$

However, since $c_i^2 \leqslant |V-c|^2$, (2.29) can be modified to read:

PROPOSITION 5. *For an inviscid flow in a flexible-walled tube with* $V_w = 0$, *all unstable axisymmetric modes have*

$$c_i < \max\left[\frac{r^2(V')^2}{4(k^2r^2+1)}\right]^{1/2}.$$

Propositions 3, 4 and 5 provide bounds on the real and imaginary parts of the phase velocity of unstable modes for an axisymmetric perturbation.

It is of interest to compare the above results with those of Yeo & Dowling (1987) and Yeo (1994) for a two-dimensional flow past a flexible surface. Propositions 1 and 2 are similar to Propositions 4 and 5 of Yeo (1994), except that the term $(V''-r^{-1}V')$ appears instead of V'' in Yeo's results. This might be anticipated, because a two-dimensional flow can be considered as the flow in a flexible tube where the variation in the mean velocity is confined to a thin region near the wall. In this case, $V'' \gg r^{-1}V'$ because the radius of the tube is large compared to the characteristic length of the mean velocity, and the results of Yeo are recovered. In the case of the Høiland theorem (Proposition 5), the present analysis requires $c_i < \max[(V')^2/4k^2]^{1/2}$, which coincides with Proposition 3 of Yeo & Dowling (1994). Similarly, Proposition 3 of the present analysis requires that $c_r < \max[V]$, which coincides with Proposition 1 of Yeo & Dowling for a monotonic velocity profile with a non-slip condition at the wall. Therefore, the present results are in agreement with those of Yeo & Dowling in the limit $k \gg 1$.

3. Non-axisymmetric perturbations

In this section, the stability of a steady mean axisymmetric flow $V(r)$ to non-axisymmetric disturbances is considered. Here, the velocity perturbation has components (v_r, v_ϕ, v_x) along the radial, azimuthal and axial directions. The normal-mode form of the velocity perturbation is

$$v_i = \tilde{v}_i(r)\exp[\mathrm{i}k(x-ct)+\mathrm{i}n\phi], \quad (3.1)$$

where n is an integer. The linearized mass and momentum equations for an inviscid flow are

$$(\mathrm{d}_r + r^{-1})\tilde{v}_r + r^{-1}\mathrm{i}n\tilde{v}_\phi + \mathrm{i}k\tilde{v}_x = 0, \tag{3.2}$$

$$\mathrm{i}k(V-c)\tilde{v}_r + \mathrm{d}_r\tilde{p} = 0, \tag{3.3}$$

$$\mathrm{i}k(V-c)\tilde{v}_\phi + \mathrm{i}nr^{-1}\tilde{p} = 0, \tag{3.4}$$

$$\mathrm{i}k(V-c)\tilde{v}_x + V'\tilde{v}_r + \mathrm{i}k\tilde{p} = 0. \tag{3.5}$$

Equations (3.2)–(3.5) can be simplified by first eliminating $\tilde{v}_\phi$ using (3.2) and (3.4). From (3.2)

$$\tilde{v}_\phi = \frac{-1}{r^{-1}\mathrm{i}n}[(\mathrm{d}_r + r^{-1})\tilde{v}_r + \mathrm{i}k\tilde{v}_x] \tag{3.6}$$

and from (3.4) and (3.5)

$$\tilde{v}_\phi = \frac{-r^{-1}\mathrm{i}n\tilde{p}}{\mathrm{i}k(V-c)} = \frac{-r^{-1}\mathrm{i}n}{k^2(V-c)}[\mathrm{i}k(V-c)\tilde{v}_x + V'\tilde{v}_r]. \tag{3.7}$$

Equating (3.6) and (3.7), we get

$$(kr)^2(V-c)[(\mathrm{d}_r + r^{-1})\tilde{v}_r + \mathrm{i}k\tilde{v}_x] + n^2[\mathrm{i}k(V-c)\tilde{v}_x + V'\tilde{v}_r] = 0. \tag{3.8}$$

Solving (3.3), (3.5) and (3.8) simultaneously, the following Rayleigh-type equation is obtained for the velocity $\tilde{v}_r$:

$$\left[\mathrm{d}_r^2 + r^{-1}\mathrm{d}_r - r^{-2}(1 + k^2r^2 + n^2) + \frac{2n^2}{k^2r^2 + n^2}\left(r^{-1}\mathrm{d}_r + r^{-2} - \frac{r^{-1}V'}{V-c}\right) - \left(\frac{V'' - r^{-1}V'}{V-c}\right)\right]\tilde{v}_r = 0. \tag{3.9}$$

It is difficult to obtain general results which are independent of the eigenvalues $\tilde{v}_r$ owing to the presence of r-dependent coefficients in this equation. However, (3.9) is simplified considerably in the limit of highly non-axisymmetric modes $n \gg k$. This corresponds to long-wavelength non-axisymmetric perturbations, for which the variation in the azimuthal direction is large compared to that in the axial direction. For these perturbations, equation (3.9) reduces to

$$\left[\mathrm{d}_r^2 + 3r^{-1}\mathrm{d}_r - r^{-2}(n^2-1) - \left(\frac{V'' + r^{-1}V'}{V-c}\right)\right]\tilde{v}_r = 0. \tag{3.10}$$

Equation (3.10) can be analysed using methods similar to those used for axisymmetric perturbations. Multiplying (3.10) by $r^3\tilde{v}_r^*$ and integrating from $r=0$ to $r=1$, we obtain the following equation:

$$\int_0^1 r^3\,\mathrm{d}r\left[|\mathrm{d}_r\tilde{v}_r|^2 + \left(r^{-2}(n^2-1) + \frac{(V'' + r^{-1}V')(V-c^*)}{|V-c|^2}\right)|\tilde{v}_r|^2\right]$$
$$= (r^3\tilde{v}_r^*\,\mathrm{d}_r\tilde{v}_r)|_{r=1} - (r^3\tilde{v}_r^*\,\mathrm{d}_r\tilde{v}_r)|_{r=0}. \tag{3.11}$$

An expression for $\tilde{v}_r^*\,\mathrm{d}_r\tilde{v}_r$ can be derived from the boundary conditions (2.9) and (2.10) which relate the velocity and pressure fields. The fluid pressure is given by (3.5):

$$\tilde{p} = -\frac{1}{\mathrm{i}k}[\mathrm{i}k(V-c)\tilde{v}_x + V'\tilde{v}_r]. \tag{3.12}$$

Equation (3.8) can be used to express $\tilde{v}_x$ in terms of $\tilde{v}_r$:

$$\mathrm{i}k(V-c)\,\tilde{v}_x + V'\tilde{v}_r = \frac{-1}{(kr)^2+n^2}[(kr)^2(V-c)\,(\mathrm{d}_r+r^{-1})+n^2V']\,\tilde{v}_r + V'\tilde{v}_r$$

$$= \frac{-(kr)^2}{(kr)^2+n^2}[(V-c)\,(\mathrm{d}_r+r^{-1})-V']\,\tilde{v}_r. \qquad (3.13)$$

Inserting this into (3.12) for the pressure field, we get

$$\mathrm{d}_r\tilde{v}_r = \frac{(n^2+(kr)^2)\,\mathrm{i}k\tilde{p}}{(kr)^2(V-c)} - r^{-1}\tilde{v}_r + \frac{V'\tilde{v}_r}{V-c}. \qquad (3.14)$$

Using (2.9) and (2.10) for the relation between the pressure and the velocity field at the wall $r=1$ (where $V=0$), the final expression for $\tilde{v}_r^*\,\mathrm{d}_r\tilde{v}_r$ is

$$(\tilde{v}_r^*\,\mathrm{d}_r\tilde{v}_r)|_{r=1} = \left[\frac{\bar{E}-k^2c^2\bar{I}-\mathrm{i}kc\bar{D}}{c^2} - 1 - \frac{V'_w}{c}\right](\tilde{v}_r\tilde{v}_r^*)|_{r=1}, \qquad (3.15)$$

where $\bar{E}$, $\bar{I}$ and $\bar{D}$ are $E(k^2+n^2)/k^2$, $I(k^2+n^2)/k^2$ and $D(k^2+n^2)/k^2$ respectively. Note that expression (3.15) for the boundary condition is similar to (2.11) for axisymmetric perturbations, since $\bar{E}$, $\bar{I}$ and $\bar{D}$ are also positive constants. Since the results derived here depend only on the sign of the boundary terms containing E, I and D, the results for the non-axisymmetric perturbations can be derived in a manner very similar to those for the axisymmetric perturbations. Also note that $\bar{E} \gg E$, and likewise for D and I, because $n \gg k$.

The equivalent of the Rayleigh theorem for a highly non-axisymmetric modes can be obtained by inserting (3.15) into equation (3.11), multiplying by c and considering the imaginary part of the resulting equation:

$$c_i\int_0^1 r^3\,\mathrm{d}r\left[|\mathrm{d}_r\tilde{v}_r|^2+|\tilde{v}_r|^2\left(r^{-2}(n^2-1)+\frac{(V''+r^{-1}V')\,V}{|V-c|^2}\right)\right]$$

$$= -\left[c_i\left(1+\frac{\bar{E}}{|c|^2}+k^2\bar{I}\right)+k\bar{D}\right](\tilde{v}_r\tilde{v}_r^*)|_{r=1}. \qquad (3.16)$$

From (3.16), the following result and its corollaries are easily deduced in a manner similar to Proposition 1 and its corollaries:

PROPOSITION 6. *For an inviscid flow in a flexible-walled tube with* $V_w=0$, *a highly non-axisymmetric mode* ($k \ll n$) *can be unstable only if* $(V''+r^{-1}V')\,V<0$ *somewhere in the flow.*

COROLLARY 1. *For an inviscid flow in a flexible-walled tube with* $V_w=0$ *and* $(V''+r^{-1}V')\,V \geqslant 0$, *all highly non-axisymmetric modes* ($k \ll n$) *are neutrally stable if the wall is non-dissipative* ($D=0$).

COROLLARY 2. *For an inviscid flow in a flexible-walled tube with* $V_w=0$ *and* $(V''+r^{-1}V')\,V \geqslant 0$, *all highly non-axisymmetric modes* ($k \ll n$) *are damped if the wall is dissipative* ($D>0$).

The equivalent of the Fjørtoft theorem for the flow in a flexible tube can be obtained by inserting (3.15) into (3.11) and taking the imaginary part of the resulting equation:

$$c_i\int_0^1 r^3\,\mathrm{d}r\left[\frac{V''+r^{-1}V'}{|V-c|^2}\tilde{v}_r\tilde{v}_r^*\right] = -\left[c_i\left(\frac{2\bar{E}c_r}{|c|^4}-\frac{V'_w}{|c|^2}\right)+\frac{k\bar{D}c_r}{|c|^2}\right](\tilde{v}_r\tilde{v}_r^*)|_{r=1}. \qquad (3.17)$$

From (3.17), the equivalent of Proposition 2 for a highly non-axisymmetric mode is:

PROPOSITION 7. *For an inviscid flow in a flexible tube with* $V_w = 0$ *and* $V'_w \leqslant 0$ ($V'_w \geqslant 0$), *an unstable highly non-axisymmetric mode* ($k \ll n$) *with* $c_r \geqslant 0$ ($c_r \leqslant 0$) *can exist only if* $(V'' + r^{-1}V') < 0$ $((V'' + r^{-1}V') > 0)$ *somewhere in the flow.*

The range of potentially unstable wavenumbers can be obtained using a derivation similar to that for Proposition 3. Equation (2.14) for f is inserted into the conservation equation (3.10), the result is multiplied by $r^3 f^*$ and integrated from $r = 0$ to $r = 1$. The equivalent of (2.16) for a highly non-axisymmetric mode, derived in this manner, is

$$\int_0^1 r^3\,\mathrm{d}r\,[(V-c)^2(|\mathrm{d}_r f|^2 + r^{-2}(n^2-1)\,|f|^2) - 2r^{-1}V'(V-c)\,|f|^2] = [(V-c)^2 f^*\,\mathrm{d}_r f]_{r=1}. \quad (3.18)$$

The right-hand side of (3.18) can be simplified to give (analogous to (2.17))

$$[(V-c)^2 f^*\,\mathrm{d}_r f]_{r=1} = [\bar{E} - k^2c^2\bar{I} - \mathrm{i}kc\bar{D} - c^2]\,(ff^*)|_{r=1}. \quad (3.19)$$

Inserting (3.19) into (3.18) and taking the imaginary part of the resulting equation, we get

$$\int_0^1 r^3\,\mathrm{d}r\,[-2c_i(V-c_r)(|\mathrm{d}_r f|^2 + r^{-2}(n^2-1)\,|f|^2) + 2r^{-1}c_i\,V'\,|f|^2]$$
$$= [-2c_r\,c_i(1+k^2\bar{I}) - kc_r\,\bar{D}]\,(ff^*)|_{r=1}. \quad (3.20)$$

The equivalent of Proposition 3 for a highly non-axisymmetric mode can be obtained from (3.20) using arguments similar to those for the axisymmetric case:

PROPOSITION 8. *For an inviscid flow in a flexible-walled tube with* $V_w = 0$ *and* $V' \leqslant 0\,(V' \geqslant 0)$ *in the tube, all unstable highly non-axisymmetric modes with* $n \gg k$ *have*

$$0 < c_r < \max\left[V - \frac{rV'}{(n^2-1)}\right] \quad \left(\min\left[V - \frac{rV'}{(n^2-1)}\right] < c_r < 0\right).$$

A bound on the magnitude of the phase velocity $|c|$ can be derived in a manner similar to Proposition 4 for axisymmetric perturbations. Equation (3.18) is multiplied by c^*, and the imaginary part of the resulting equation is

$$\int_0^1 r^3\,\mathrm{d}r\,[c_i\,[(|c|^2 - V^2)(|\mathrm{d}_r f|^2 + r^{-2}(n^2-1)\,|f|^2) + 2r^{-1}c_i\,V'\,V\,|f|^2]$$
$$= -[c_i\,\bar{E} + c_i(k^2\bar{I}+1)\,|c|^2 + k|c|^2\bar{D}]\,(ff^*)|_{r=1}. \quad (3.21)$$

The right-hand side of (3.21) is always negative for unstable modes. Using arguments similar to those for Proposition 4, we obtain

PROPOSITION 9. *For an inviscid flow in a flexible-walled tube with* $V_w = 0$ *in the tube, all unstable highly non-axisymmetric modes with* $n \gg k$ *have*

$$|c|^2 < \max\left[V^2 - \frac{2rVV')}{(n^2-1)}\right].$$

The equivalent of Proposition 5 for the present case can be derived as follows.

Equation (2.22), which defines the function g, is inserted into the conservation equation (3.10), multiplied by r^3g^* and integrated from $r = 0$ to $r = 1$ to give

$$\int_0^1 r^3\,\mathrm{d}r\left[(V-c)\,(|\mathrm{d}_r g|^2+r^{-2}(n^2-1)\,|g|^2)\right.$$
$$\left.+\left(\frac{V''}{2}+\frac{(V')^2}{4(V-c)}-\frac{3r^{-1}V'}{2}\right)|g|^2\right] = (g^*(V-c)\,\mathrm{d}_r g)|_{r=1}. \quad (3.22)$$

The right-hand side of (3.22) can be simplified using (3.14) for $\mathrm{d}_r\tilde{v}_r$:

$$(g^*(V-c)\,\mathrm{d}_r g)|_{r=1} = \left[-\frac{\bar{E}-c^2(1+k^2\bar{T})-\mathrm{i}kc\bar{D}}{c}+\frac{V'}{2}\right](gg^*)|_{r=1}. \quad (3.23)$$

Inserting (3.23) into (3.22), taking the imaginary part, and multiplying the result by c_i, we get

$$\int_0^1 r^3\,\mathrm{d}r\left[-c_i^2(|\mathrm{d}_r g|^2+r^{-2}(n^2-1)\,|g|^2)+\frac{c_i^2(V')^2\,|g|^2}{4\,|V-c|^2}\right]$$
$$=\left[\frac{c_i^2\,\bar{E}+|c|^2c_i^2(k^2\bar{T}+1)+k|c|^2c_i\,\bar{D}}{|c|^2}\right](gg^*)|_{r=1}. \quad (3.24)$$

The right-hand side of (3.24) is always positive. A necessary condition for the left-hand side to be positive can be obtained using a derivation similar to that for Proposition 5:

$$c_i^2(n^2-1) < \max\left(\frac{r^2(V'^2)\,c_i^2}{4|V-c|^2}\right). \quad (3.25)$$

Using $c_i^2 \leqslant |V-c|^2$, we obtain the following equivalent of the Høiland theorem:

PROPOSITION 10. *For an inviscid flow in a flexible-walled tube with* $V_w = 0$ *in the tube, all unstable highly non-axisymmetric modes with* $n \gg k$ *have*

$$c_i < \max\left[\frac{r^2(V')^2}{4(n^2-1)}\right]^{1/2}.$$

4. Conclusions

In the present analysis, results for the possibility of unstable modes and the range of the phase velocity and growth rate of these modes were obtained for a general velocity profile in a flexible tube. These results are valid for an axisymmetric mean flow which satisfies the no-slip condition at the wall, and varies monotonically towards the centre of the tube. This is not a severe restriction, however, because most practical flows such as entry flows in cylindrical tubes and flows in tubes of slowly varying cross-section satisfy these conditions. Unlike the case of plane two-dimensional channels, there is no equivalent of the Squire transformation, which shows that non-axisymmetric perturbations are more stable than axisymmetric ones, in a cylindrical geometry. Therefore, it is necessary to analyse both axisymmetric and non-axisymmetric perturbations to determine the stability of the flow.

The perturbation to the mean velocity field was considered to be of the form $v_i = \tilde{v}_i(r)\exp[\mathrm{i}k(x-ct)+\mathrm{i}n\phi]$, and results were derived for two limiting cases: axisymmetric perturbations ($n = 0$), and highly non-axisymmetric perturbations

($n \gg k$) where the variation in the polar direction is large compared to that in the axial direction. The main results are summarized in table 1. The results for both axisymmetric and highly non-axisymmetric perturbations are similar to the classical theorems of hydrodynamic stability, and using these it should be possible to determine whether a mean flow could be unstable. In addition, bounds are derived for the real and imaginary parts of the phase velocity in both limits. The bounds prove inconclusive for the case $n = 1$ for highly non-axisymmetric perturbations, but the range of variation of the phase velocity becomes smaller as n increases. In particular, the growth rate c_i decreases proportional to n^{-2} in the limit $n \gg 1$, indicating that the higher harmonics become more stable as n increases beyond 1.

Proposition 1 and its corollaries indicate that for the practically important case of steady Hagen–Poiseuille flow for which $V'' - r^{-1}V' = 0$, perturbations are always neutrally stable for non-dissipative surfaces. This is in agreement with an earlier study (Kumaran 1995*b*) of the high Reynolds number flow through a flexible tube. In the leading-order approximation, the viscous effects were neglected in the fluid and wall material and the flow was considered to be an inviscid flow through an elastic tube. There were multiple solutions for the growth rate of the perturbations, which were all imaginary indicating that the perturbations are neutrally stable, as predicted by Proposition 1. This result is not conclusive, however, and it is necessary to determine the effect of the viscous correction to the leading-order flow to determine the stability of the flow. This was examined in Kumaran (1995*b*), and it was found that the correction to the growth rate is $O(Re^{-1/2})$ smaller than the leading-order frequency, due to the presence of a wall layer of thickness $O(Re^{-1/2})$ at the wall. There is a transfer of energy from the mean flow to the fluctuations due to the convective (Reynolds stress) terms in the conservation equation, and due to the shear work down by the mean flow at the surface. These two energy transfer mechanisms turn out to have the same magnitude and opposite directions, resulting in no net transfer of energy from the mean flow to the fluctuations, and the fluctuations are stabilized due to the viscous dissipation in the wall layer. There are certain parameter values, however, where the wall layer amplitude becomes zero because the tangential velocity boundary condition is identically satisfied by the inviscid flow solutions. At these parameter values, the $O(Re^{-1/2})$ correction to the growth rate is zero, and the $O(Re^{-1})$ correction to the growth rate is negative, indicating that there is a small stabilizing effect due to the dissipation in the bulk of the fluid and the wall.

Another situation of practical interest is the flow in the entrance region of a tube, where there is a transition from a plug flow to fully developed Hagen–Poiseuille flow, and it is of interest to examine whether the velocity profile in the entrance region could become unstable to axisymmetric perturbations. There are no general analytical solutions for the flow in the entrance region, but very near the entrance of the tube the velocity profile is similar to the boundary layer flow over a flat plate. In this region, a perturbation solution in the small parameter $\epsilon = \nu x/R^2 V_0$ can be obtained using the method of Schlichting (1934, 1955). Here, V_0 is the uniform velocity at the entrance of the tube, x is the distance from the entrance of the tube and ν is the kinematic velocity. The solution is of the form

$$V(r) = V_0(f_1'(\eta) + \epsilon f_2'(\eta) + \ldots), \tag{4.1}$$

where $\eta \equiv [y/(x\nu/V_0)^{1/2}]$ is the Blasius similarity variable and primes represent derivatives with respect to η. In (4.1), $f_1(\eta)$ is the Blasius stream function (Batchelor 1967) which is obtained by solving

$$f_1''' + \tfrac{1}{2} f_1 f_1'' = 0 \tag{4.2}$$

	Axisymmetric perturbations	Highly non-axisymmetric perturbations
Velocity profiles that are always stable	$(V''-r^{-1}V')\,V \geqslant 0$	$(V''+r^{-1}V')\,V \geqslant 0$
Range of c_r for unstable modes	$0 < c_r < \max\left[V - \dfrac{rV'}{(k^2r^2+1)}\right]$	$0 < c_r < \max\left[V - \dfrac{rV'}{n^2-1}\right]$
Range of $\|c\|^2$ for unstable modes	$\|c\|^2 < \max\left[V^2 - \dfrac{2rVV'}{k^2r^2+1}\right]$	$\|c\|^2 < \max\left[V^2 - \dfrac{2rVV'}{n^2-1}\right]$
Range of c_i for unstable modes	$c_i < \max\left[\dfrac{r^2(V')^2}{4(k^2r^2+1)}\right]^{1/2}$	$c_i < \max\left[\dfrac{r^2(V')^2}{4(n^2-1)}\right]^{1/2}$

TABLE 1. Stable velocity profiles and bounds on the phase velocity of unstable modes for axisymmetric ($n = 0$) and highly non-axisymmetric ($n \gg k$) perturbations

subject the boundary conditions

$$\left.\begin{aligned} f_1(\eta) &= 0 \quad \text{at} \quad \eta = 0, \\ f_1'(\eta) &= 0 \quad \text{at} \quad \eta = 0, \\ f_1'(\eta) &= 1 \quad \text{at} \quad \eta \to \infty, \end{aligned}\right\} \tag{4.3}$$

and the higher-order corrections $f_2, f_3, \ldots$, are obtained by solving the higher-order equations for the stream function. In the present geometry, the distance from the wall $y = R-r$, and the limit $\epsilon \ll 1$ implies $R \gg (\nu x/V_0)^{1/2}$. Very near the entrance of the tube, the thickness of the boundary layer is small compared to the radius of the tube and the velocity profile is identical to that for the Blasius boundary layer over a flat plate. For this velocity profile, the possibility of unstable modes depends on the sign of

$$V'' - r^{-1}V' = V_0\left[\frac{V_0 f_1'''}{\nu x} + R^{-1}\left(\frac{V_0}{\nu x}\right)^{1/2} f_1''\right]. \tag{4.4}$$

In the limit $\epsilon \ll 1$, the second term on the right-hand side is $O(\epsilon^{1/2})$ smaller than the first, and can be neglected. The solution for the Blasius steam function $f_1(\eta)$ is well known (Batchelor 1967), and the third derivative f_1''' is negative in the boundary layer, indicating that $(V''-r^{-1}V')\,V$ is negative near the entrance of the tube. Therefore, the simple calculation given above indicates that the velocity profile near the entrance of the tube could be unstable to inviscid axisymmetric perturbations according to Proposition 1.

For highly non-axisymmetric perturbations, Proposition 6 indicates that a Hagen–Poiseuille flow could become unstable, since $(V''+r^{-1}V')\,V$ is negative for this flow. This is in contrast to parallel flows, where three-dimensional perturbations are always more stable than two-dimensional ones. Coupled with the predictions of Propositions 8, 9 and 10 that the bound on the growth rate decreases as n increases beyond 1, this indicates that non-axisymmetric modes with finite n are the most unstable modes in a flexible tube.

Numerical studies of the air flow through a tube with a compliant lining were carried out by Evrensel *et al.* (1993), following the experimental work of King *et al.* (1985) on the air flow past a layer of mucus. The numerical studies used a tube diameter of 0.74 cm, and the shear wave speed of the mucus layer, $C_t = (G/\rho_m)^{1/2}$, was 4.47 cm s^{-1}, where G and ρ_m are the shear modulus and density of the mucus.

Both elastic and viscoelastic surfaces were examined by Evrensel *et al.*, and they found very different critical velocities in the two cases. For elastic surfaces, they found that the axisymmetric disturbances become unstable at an air velocity V of about 28 cm s^{-1}. However, at these velocities, the Reynolds number based on the tube diameter and the density and viscosity of air is only about 150, which may be too low to justify the inviscid flow approximation. The studies on a viscoelastic tube wall, however, found a critical velocity of about 1045 cm s^{-1}. The Reynolds number in this case is about 5500, and the inviscid approximation could be valid in this case. However, the value of V/C_t is $O(234)$, indicating that the unstable modes are not inviscid modes, for which $V/C_t \sim 1$ (see Kumaran 1995*b*), but the instability could be induced by viscous effects.

Proposition 6 states that a Hagen–Poiseuille flow could be unstable to non-axisymmetric modes, since $(V'' + r^{-1}V')\,V < 0$ for a parabolic flow. It is useful to examine the parameter ranges in which this instability might be expected. From (3.17), (3.21) and (3.24), it can be seen that an instability can only be observed if the effective stiffness of the wall is $E/\rho V_m^2 \sim O(k^2/n^2)$. Since $k \ll n$ in the highly non-axisymmetric limit, this condition is equivalent to $E/\rho V_m^2 \ll 1$. In air flows through soft materials such as the pulmonary linings, this condition is only satisfied for $V_m \gg 100$ cm s^{-1}, since the shear modulus of these materials is $O(10$ dyn $cm^{-2})$ and the density of air is $O(10^{-3}$ g $cm^{-3})$. Therefore, non-axisymmetric inviscid modes would be observed only at velocities of 10^3–10^4 cm s^{-1} ($Re = 10^3$–10^4), and viscosity-induced axisymmetric modes could appear at lower velocities, as observed by Evrensel *et al.* However, for the flow of liquids in flexible tubes, the condition is satisfied for $V_m \gg 3$ cm s^{-1} since the density of liquids is $O(1$ g $cm^{-3})$. In this case, non-axisymmetric modes could become unstable at velocities of 10–100 cm s^{-1} ($Re = 10^2$–10^3). Since the velocity is relatively low in this case, non-axisymmetric disturbances could appear at lower velocities than viscosity-induced axisymmetric disturbances, and the results obtained here could be relevant for liquid flows in flexible tubes.

REFERENCES

BATCHELOR, G. K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press.

BENJAMIN, T. B. 1959 Shearing flow over a wavy boundary layer. *J. Fluid Mech.* **6**, 161–205.

BENJAMIN, T. B. 1963 The threefold classification of unstable disturbances in flexible surfaces bounding inviscid flows. *J. Fluid Mech.* **16**, 436–450.

CARPENTER, P. W. 1990 Status of transition delay using compliant walls. *Prog. Astro. Aero.* **123**, 79–113.

CARPENTER, P. W. & GAJJAR, J. G. S. 1990 A general theory for two- and three-dimensional wall mode instabilities in boundary layers over isotropic and anisotropic compliant walls. *Theor. Comput. Fluid Dyn.* **1**, 349–378.

CARPENTER, P. W. & GARRAD, A. D. 1985 The hydrodynamic stability of flows over Kramer-type compliant surfaces. Part 1. Tollmien–Schlichting instabilities. *J. Fluid Mech.* **155**, 465–510.

CARPENTER, P. W. & GARRAD, A. D. 1986 The hydrodynamic stability of flows over Kramer-type compliant surfaces. Part 2. Flow induced surface instabilities. *J. Fluid Mech.* **170**, 199–232.

CORCOS, G. M. & SELLARS, J. R. 1959 On the stability of fully developed flow in a pipe. *J. Fluid Mech.* **5**, 97–112.

DAVEY, A. & DRAZIN, P. G. 1969 The stability of Poiseuille flow in a pipe. *J. Fluid Mech.* **36**, 209–218.

DRAZIN, P. G. & REID, W. H. 1981 *Hydrodynamic Stability*. Cambridge University Press.

EVRENSEL, C. A., KHAN, M. R. U., ELLI, S. & KRUMPE, P. E. 1993 Viscous airflow through a rigid tube with a compliant lining: a simple model for the air-mucus interaction in pulmonary airways. *Trans. ASME* K: *J. Biomech. Engng* **115**, 262–270.

GAD-EL-HAK, M., BLACKWELDER, R. F. & RILEY, J. J. 1984 On the interaction of compliant coatings with boundary-layer flows. *J. Fluid Mech.* **140**, 257–280.

GARG, V. K. & ROULEAU, W. T. 1972 Linear spatial stability of pipe Poiseuille flow. *J. Fluid Mech.* **54**, 113–127.

GILL, A. E. 1965 On the behaviour of small disturbances to Poiseuille flow in a circular pipe. *J. Fluid Mech.* **21**, 145–172.

HANSEN, R. J., HUNSTON, D. L., NI, C. C. & REISCHMANN, M. M. 1980 An experimental study of flow-generated waves on a flexible surface. *J. Sound. Vib.* **68**, 317–325.

KING, M., BROCK, G. & LUNDELL, C. 1985 Clearance of mucus by simulated cough. *J. Appl. Physiol.* **58**, 1172–1180.

KRAMER, M. O. 1960 Boundary layer stabilization by distributed damping. *J. Am. Soc. Naval Engrs* **74**, 25.

KRINDEL, P. & SILBERBERG, A. 1979 Flow through gel-walled tubes. *J. Colloid Interface Sci.* **71**, 34–50.

KUMARAN, V. 1995*a* Stability of the viscous flow of a fluid through a flexible tube. *J. Fluid Mech.* **294**, 259–281.

KUMARAN, V. 1995*b* Stability of the flow of a fluid through a flexible tube at high Reynolds number. *J. Fluid Mech.* **302**, 117–139.

LANDAHL, M. T. 1962 On the stability of a laminar incompressible boundary layer over flexible surface. *J. Fluid Mech.* **13**, 609–632.

ROTENBERRY, J. M. & SAFFMAN, P. G. 1990 Effect of compliant boundaries on the weakly nonlinear shear waves in a channel flow. *SIAM J. Appl. Maths* **50**, 361–394.

SALWEN, H. & GROSCH, C. E. 1972 The stability of Poiseuille flow in a pipe of circular cross section. *J. Fluid Mech.* **54**, 93–112.

SCHLICHTING, H. 1934 Laminare Kanaleinlaufstromung. *Z. Agnew. Math. Mech.* **14**, 368–373.

SCHLICHTING, H. 1955 *Boundary Layer Theory*. McGraw Hill.

TOLLMIEN, W. 1929 Uber die Entstehung der Turbulenz. 1. *Mitt., Ges. Wiss. Gottingen, Math. Phys. Klasse* 21–44.

WYGNANSKI, I. & CHAMPAGNE, F. H. 1973 On transition in a pipe. Part 1. The origin of puffs and slugs, and the flow in a turbulent slug. *J. Fluid Mech.* **59**, 281–335.

YEO, K. S. 1994 Note on the inviscid stability of flow over a compliant wall. *J. Fluid Mech.* **279**, 165–168.

YEO, K. S. & DOWLING, A. P. 1987 The stability of inviscid flows over passive compliant walls. *J. Fluid Mech.* **183**, 265–292.