Chapter 3

Transport in one and two dimensions

In this chapter, we consider transport in which there is a variation in the mass, momentum and temperature fields in one dimension. The analysis is considerably simplified in this case, since there is variation with respect to only one spatial co-ordinate, in addition to variation in time for unsteady diffusion and flow problems. However, the examples solved here illustrate the basic principles of the solution of more complex problems in multiple dimensions, which involve shell balances to derive differential equations for the concentration, velocity and temperature fields, and then an integration procedure for determining the variations in the concentration, velocity and temperature.

3.1 Transport between flat plates:

The simplest configuration consists of two plates of infinite extent separated by length L, as shown in figure **??**. There is a flux of mass, momentum or energy due to a difference in the concentration, velocity or temperature between the two plates. At steady state, there is a constant flux between the two plates in the direction perpendicular to the two plates, and the concentration, velocity or temperature varies linearly with the distance.

One could also consider an unsteady situation, where the entire system is at a temperature T_0, and the temperature of one plate is instanteneously changed to T_1, as shown in figure **??**. The equivalent situations in mass and momentum transfer involves a system initially at constant concentration c_0, in which the concentration of one plate is instanteneously set equal to c_1. In momentum transfer, one could consider a system which is initially stationary, in which the velocity of one surface is set equal to U. The temperature, concentration or velocity is initially a step function, and then it evolves in time and finally approaches a steady linear temperature profile in the long time limit.

In this chapter, we will also consider situations involving periodic oscillations in the temperature, concentration or velocity on one plate. In the case of momentum transport, this involves one stationary plate and another plate oscillating at a fixed frequency. Similar situations can be considered for heat and mass transport. In all cases, we first derive equations for the variation, both in position and time, of the concentration, temperature or velocity in between the two plates. When expressed in terms of the scaled concentration, temperature or velocity fields, these equations are identical in form. The solution procedures for these equations under steady and unsteady situations are then discussed.

3.2 Cartesian co-ordinates:

3.2.1 Mass transfer

Consider two flat surfaces in the $x - y$ plane, separated by a distance L, located at $z = 0$ and $z = L$. through which the solvent diffuses into the fluid, as shown in figure reffig311. The temperature is c_0 at the top plate, and c_1 at the bottom plate. An equation for the variation of concentration with z and with time can be derived from the mass conservation condition,

Consider a shell of thickness Δz in the z coordinate as shown in figure 3.1, and of area $\Delta x \Delta y$ in the $x-y$ plane. There is a transport of mass across the surfaces of the shell due to diffusion, which results in a change in the concentration in the shell. We consider the variation in the concentration within this control volume over a time interval Δt. Mass conservation requires that

$$\left(\begin{array}{c} \text{Accumulation of} \\ \text{mass in the shell} \end{array} \right) = \left(\begin{array}{c} \text{Input of} \\ \text{mass into shell} \end{array} \right) - \left(\begin{array}{c} \text{Output of} \\ \text{mass from shell} \end{array} \right) + \left(\begin{array}{c} \text{Production of} \\ \text{mass in shell} \end{array} \right) \tag{3.1}$$

The accumulation of mass in a time Δt is given by

$$\begin{pmatrix} \text{Accumulation of mass} \\ \text{in the shell} \end{pmatrix} = (c(x,y,z,t+\Delta t) - c(x,y,x,t))\Delta x \Delta y \Delta z \quad (3.2)$$

The transport of mass takes place due to molecular diffusion only in the z direction, because there is concentration variation only in this direction. Therefore, we need to consider the transport across the two surfaces located at z and $z + Deltaz$. The total mass entering the shell through the surface at z in a time interval Δt is given by the product of the mass flux, the area of transfer $\Delta x \Delta y$ and the time interval Δt,

$$\begin{pmatrix} \text{Input of} \\ \text{mass into shell} \end{pmatrix} = j_z|_z \, \Delta t \Delta x \Delta y \quad (3.3)$$

In a similar manner, the mass leaving the surface at $z + \Delta z$ is given by

$$\begin{pmatrix} \text{Output of} \\ \text{mass from shell} \end{pmatrix} = j_z|_{z+\Delta z} \, \Delta t \Delta x \Delta y \quad (3.4)$$

There could also be a rate of production (or consumption) of mass in the shell due to a chemical reaction. This term is positive if c is the concentration of a species produced in the reaction, while it is negative if c is the concentration of a species consumed in the reaction. The mass produced in the volume $\Delta x \Delta y \Delta z$ within the time Δt is $S \Delta x \Delta y \Delta z \Delta t$, where S is the rate of production of mass per unit volume per unit time. This rate of reaction is a function of the concentrations of the reacting species and the temperature, which are functions of position and time, and so S could depend on position and time. However, since we are considering variations only in the z direction and time, the production rate is assumed to be a function of z and t.

$$\begin{pmatrix} \text{Production of} \\ \text{mass in shell} \end{pmatrix} = S(z,t) \Delta x \Delta y \Delta z \Delta t \quad (3.5)$$

Substituting equations 3.2, 3.3 3.4 and 3.5 into equation 3.1, and dividing by $\Delta x \Delta y \Delta z \Delta t$, we obtain

$$\frac{c(x,y,z,t+\Delta t) - c(x,y.z,t)}{\Delta t} = \frac{1}{\Delta z}\left(j_z|_z - j_z|_{z+\Delta z}\right) + S(z,t) \quad (3.6)$$

The above equation is a 'difference' equation, since it relates the difference in the concentration at two different locations and times. This difference equation can be converted into a differential equation by taking the limit $\Delta t \to 0$ and $\Delta z \to 0$.

$$\frac{\partial c}{\partial t} = -\frac{\partial j_z}{\partial z} + S \tag{3.7}$$

Using the Fick's law for diffusion,

$$j_z = -D\frac{\partial c}{\partial z} \tag{3.8}$$

the concentration diffusion equation 3.7 becomes,

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial z}\left(D\frac{\partial c}{\partial z}\right) + S \tag{3.9}$$

The above equation is a 'partial differential equation', since it contains derivatives with respect to two independent variables, z and t. (This is in contrast to an 'ordinary differential equation', which contains derivatives with respect to only one independent variable).

If the diffusion coefficient is a constant (a good approximation in most cases of practical interest), the differential equation 3.9 reduces to

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial z^2} + S \tag{3.10}$$

3.2.2 Heat transfer:

The equivalent heat transfer problem involves two plates of temperature T_0 at $z = L$, and temperature T_1 at $z = 0$. A shell of of thickness Δz in the z coordinate, and of area $\Delta x \Delta y$ in the $x - y$ plane, as shown in figure 3.1, is considered. The energy conservation condition is

$$\left(\begin{array}{c}\text{Accumulation of energy}\\ \text{in the shell}\end{array}\right) = \left(\begin{array}{c}\text{Input of}\\ \text{energy into shell}\end{array}\right) - \left(\begin{array}{c}\text{Output of}\\ \text{energy from shell}\end{array}\right) + \left(\begin{array}{c}\text{Production}\\ \text{energy in sh}\end{array}\right. \tag{3.11}$$

The accumulation of mass in a time Δt is given by

$$\left(\begin{array}{c}\text{Accumulation of energy}\\ \text{in the shell}\end{array}\right) = (\rho C_p(T(x,y,z,t+\Delta t) - T(x,y,x,t)))\Delta x \Delta y \Delta z \tag{3.12}$$

The total heat entering the shell through the surface at z in a time interval Δt is given by the product of the heat flux, the area of transfer $\Delta x \Delta y$ and the time interval Δt,

$$\begin{pmatrix} \text{Input of} \\ \text{heat into shell} \end{pmatrix} = q_z|_z \, \Delta x \Delta y \Delta t \tag{3.13}$$

In a similar manner, the heat leaving the surface at $z + \Delta z$ is given by

$$\begin{pmatrix} \text{Output of} \\ \text{heat from shell} \end{pmatrix} = q_z|_{z+\Delta z} \, \Delta x \Delta y \Delta t \tag{3.14}$$

The production (or consumption) of heat in the shell due to several reasons, such chemical reaction (exothermic or endothermic), heat of dissolution, latent heat due to phase transformations, or even due to viscous heating. The heat produced in the volume $\Delta x \Delta y \Delta z$ within the time Δt is $S_e \Delta x \Delta y \Delta z \Delta t$, where S_e is the rate of production of heat per unit volume per unit time. As in the case of production of mass, we assume this is a function of z and time,

$$\begin{pmatrix} \text{Production of} \\ \text{mass in shell} \end{pmatrix} = S_e(z,t) \Delta x \Delta y \Delta z \Delta t \tag{3.15}$$

Substituting equations 3.12, 3.13 3.14 and 3.15 into equation 3.11, and dividing by $\Delta x \Delta y \Delta z \Delta t$, we obtain

$$\frac{\rho C_p (T(x,y,z,t+\Delta t) - T(x,y,z,t))}{\Delta t} = \frac{q_z|_z - q_z|_{z+\Delta z}}{\Delta z} + S_e \tag{3.16}$$

The above equation is a ‘difference’ equation, since it relates the difference in the concentration at two different locations and times. This difference equation can be converted into a differential equation by taking the limit $\Delta t \to 0$ and $\Delta z \to 0$.

$$\rho C_p \frac{\partial T}{\partial t} = -\frac{\partial q_z}{\partial z} + S_e \tag{3.17}$$

Using Fourier’s law for heat conduction,

$$q_z = -k \frac{\partial T}{\partial z} \tag{3.18}$$

the energy conservation equation can be written as,

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z} \right) + S_e \tag{3.19}$$

If the thermal conductivity is independent of the z co-ordinate, the energy conservation equation can be written as,

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right) + \frac{S_e}{\rho C_p} \tag{3.20}$$

where $\alpha = (k/\rho C_p)$ is the thermal diffusivity.

3.2.3 Momentum transport:

Though the final equation for the momentum transfer process is identical to equations 3.10 and 3.20, the procedure is slightly different, and so we provide a brief outline of the calculation. First, note that there are now two directions in the problem. Since momemtum is a vector, there is a direction associated with the momentum itself. In the present problem, this is the x direction, because the velocity of the fluid is in the x direction. The second is the direction of variation of the momentum, which is the z direction in this problem, because the fluid velocity is varying only in the z direction. Since diffusion takes place along the direction where there is a variation of momentum, the diffusion in the present problem is also in the z direction.

The momentum balance equation (Newton's third law), equivalent of equation 3.1 and 3.11, is

$$\left(\begin{array}{c}\text{Rate of change of}\\ x\text{ momentum}\\ \text{in the shell}\end{array}\right) = \left(\begin{array}{c}\text{Surface of forces}\\ \text{in } x \text{ direction}\end{array}\right) + \left(\begin{array}{c}\text{Body forces}\\ \text{in } x \text{ direction}\end{array}\right) \tag{3.21}$$

The total fluid mass in the differential volume is $\rho \Delta x \Delta y \Delta z$, where ρ is the fluid density and the volume of fluid is $\Delta x \Delta y \Delta z$. We assume that the density is a constant, so that the change in momentum (mass times velocity) is due to the change in the velocity. The rate of change of momentum (change in momentum per unit time) in the differential volume of thickness Δz about z in a time interval Δt is given by,

$$\left(\begin{array}{c}\text{Rate of change of}\\ x\text{ momentum}\\ \text{in the shell}\end{array}\right) = \frac{(rho \Delta x \Delta y \Delta z)(u_x(x,y,z,t+\Delta t) - u_x(x,y,z,t)}{\Delta t} \tag{3.22}$$

The forces acting are of two types. The first is the 'body force', such as the gravitational, centrifugal and other forces, which act throughout the body. The second is the 'surface force' acting on the bounding surfaces, pressure and the shear stress. The body forces (centrifugal, gravitational, etc.) can be written as,

$$\begin{pmatrix} \text{Body forces} \\ \text{in } x \text{ direction} \end{pmatrix} = f_x \Delta x \Delta y \Delta z \tag{3.23}$$

where f_x is the force per unit volume in the x direction. The two important body forces we will encounter are the gravitational force $f_x = \rho g_x$, where ρ is the mass density (mass per unit volume) and g_x is the component of the acceleration due to gravity in the x direction, and the centrifugal force, $f_x = \rho \Omega^2 r$, where Ω is the angular velocity and r is the distance from the axis of rotation.

The surfaces forces acting on the two surfaces at z and $z + \Delta z$ are the products of the shear stress τ_{xz} and the surface area $(\Delta x \Delta y)$. It is important to keep account of the directions of the forces in this case, since the force is a vector. The shear stress τ_{xz} is defined as the force per unit area in the x direction acting at a surface whose *outward* unit normal is in the positive z direction. For the surface at $z + \Delta z$, the outward unit normal is in the $+z$ direction, as shown in figure 3.1, and therefore the force per unit area at this surface is $+ \left.\tau_{xz}\right|_{z+\Delta z}$. For the surface at z, the outward unit normal is in the $-z$ direction, and therefore the force per unit area at this surface is $- \left.\tau_{xz}\right|_{z}$. Therefore,

$$\begin{pmatrix} \text{Surface of forces} \\ \text{in } x \text{ direction} \end{pmatrix} = \Delta y \Delta z (\tau_{xz}(z + \Delta z, t) - \tau_{xz}(z, t)) \tag{3.24}$$

Therefore, the momentum balance equation is,

$$(\Delta x \Delta y \Delta z) \frac{\rho \Delta u_x}{\Delta t} = \Delta x \Delta y (\left.\tau_{xz}\right|_{z+\Delta z} - \left.\tau_{xz}\right|_{z}) + f_x \Delta x \Delta y \Delta z \tag{3.25}$$

Dividing throughout by $A \Delta z$, we obtain,

$$\rho \frac{\Delta u_x}{\Delta t} = \frac{\left.\tau_{xz}\right|_{z+\Delta z} - \left.\tau_{xz}\right|_{z}}{\Delta z} + f_x \tag{3.26}$$

Taking the limit $\Delta t \to 0$ and $\Delta z \to 0$, we obtain the partial differential equation,

$$\rho \frac{\partial u_x}{\partial t} = \frac{\partial \tau_{xz}}{\partial z} + f_x \tag{3.27}$$

Note that f is a 'force density', which is the force acting per unit volume.

The shear stress is given by the product of the viscosity and the gradient of the velocity,

$$\tau_{xz} = \mu \frac{\partial u_x}{\partial z} \tag{3.28}$$

With this, the governing equation for the velocity field becomes,

$$\rho \frac{\partial u_x}{\partial t} = \frac{\partial}{\partial z} \left(\mu \frac{\partial u_x}{\partial z} \right) + f_x \tag{3.29}$$

The differential equation derived above has the same form as the concentration and energy diffusion equations 3.1 and 3.11, though it was derived from a force balance. This shows that the diffusion process is the same for mass, momentum and energy. However, it should be noted that momentum could be transmitted by pressure forces in addition to viscous forces, and there is no analogue of pressure in mass and energy transport.

The momentum conservation equation can be recast in terms of the momentum diffusivity ν, if the viscosity and density are constants,

$$\frac{\partial u_x}{\partial t} = \nu \left(\frac{\partial^2 u_x}{\partial z^2} \right) + (f_x/\rho) \tag{3.30}$$

where $\nu = (\mu/rho)$ is the momentum diffusivity.

3.2.4 Steady and unsteady solutions:

We now solve the diffusion equation in a sequence of problems increasing in complexity, starting from the steady solution, and then moving on to the unsteady solution in an infinite domain, the unsteady solution in a finite domain, and finally a solution that is oscillatory in time. After this, we consider the effect of sources of mass and energy, as well as body forces exerted on the fluid.

At steady state, we solve the equations, for mass, momentum and energy conservation of the form,

$$\frac{\partial^2 c}{\partial z^2} = 0; \frac{\partial^2 T}{\partial z^2} = 0; \frac{\partial^2 u_x}{\partial z^2} = 0 \tag{3.31}$$

with boundary conditions,

$$c = c_1; T = T_1; u_x = U \qquad \text{at} z = 0 \tag{3.32}$$

$$c = c_0; T = T_0; u_x = U \qquad \text{at} z = H \tag{3.33}$$

It is a good practice to first non-dimensionalise the co-ordinate z, as well as the concentration, temperature and velocity. For this problem, it is appropriate to use the scaling $z^* = (z/H)$, so that z^* varies between 0 and 1 in the domain between the plates. The scaled concentration, temperature and velocity can be defined as,

$$c^* = \frac{c - c_0}{c_1 - c_0} \tag{3.34}$$

$$T^* = \frac{T - T_0}{T_1 - T_0} \tag{3.35}$$

$$u_x^* = \frac{u_x}{U} \tag{3.36}$$

When scaled in this manner, the boundary conditions for the mass, momentum and energy transport problems are identical,

$$c^* = T^* = u_x^* = 1 \text{ at } z^* = 0 \tag{3.37}$$

$$c^* = T^* = u_x^* = 0 \text{ at } z^* = 1 \tag{3.38}$$

It is quite easy to obtain the linear solutions for the concentration, temperature and velocity equations, 3.31, which satisfy the boundary conditions 3.38,

$$c^* = T^* = u_x^* = 1 - z^* \tag{3.39}$$

As expected, the concentration, temperature and velocity profiles are linear because the fluxes are constant.

3.2.5 Unsteady transport into an infinite fluid:

Let us now consider the unsteady state transport of mass/momentum/heat in a fluid between two flat plates, as shown in figure **??**, with no sources. In the mass transfer problem, the fluid and both plates are initially at a concentration c_0. At time $t = 0$, the temperature of the lower plate is instanteneously set equal to $c_1 > c_0$. There is a heat flux from the bottom plate, and the temperature increases upwards. In the final steady state, the linear concentration profile equation 3.39 is obtained. Here, we shall be concerned with the very initial stages, when the 'penetration depth' from the bottom surface is small compared to the distance between the two plates, L. In this case, the we can consider the region near the bottom plate alone, and consider the fluid to be of infinite extent in the z direction. Instead of the boundary condition equation 3.38, we can use the boundary conditions $c = c_1$ at $z = 0$ and $c = c_0$ as $z \to \infty$. The initial condition is $c = c_0$ for all $z > 0$ at $t = 0$. The mass diffusion equation is 3.31, and the boundary and initial conditions are,

$$c = 0 \qquad \text{as } z \to \infty \text{ at all } t \tag{3.40}$$

$$c = c_0 \qquad \text{at } z = 0 \text{ at all } t > 0 \tag{3.41}$$

$$c = 0 \qquad \text{at } t = 0 \text{ for all } z > 0 \tag{3.42}$$

The scaled concentration field c^* is defined in equation 3.34, and the conditions for c^* are

$$c^* = 0 \qquad \text{as } z \to \infty \text{ at all } t \tag{3.43}$$

$$c^* = 1 \qquad \text{at } z = 0 \text{ at all } t > 0 \tag{3.44}$$

$$c^* = 0 \qquad \text{at } t = 0 \text{ for all } z > 0 \tag{3.45}$$

The diffusion equation for the concentration field is,

$$\frac{\partial c^*}{\partial t} = D\frac{\partial^2 c^*}{\partial z^2} \tag{3.46}$$

In the equivalent heat and momentum transfer problems, we substitute T^* and u_x^* instead of c^*, and the thermal diffusivity α and kinematic viscosity ν instead of the mass diffusivity D.

In order to solve the concentration equation 3.46 with the boundary and initial conditions 3.45, it is first important to realise that there no intrinsic length scale in the problem, because the boundary conditions are applied at $z^* = 0$ and $z^* \to \infty$. Since the concentration c^* is dimensionless, there are only three dimensional variables z, t and D in the problem. These contain two dimensions, $\mathcal{L}$ and $\mathcal{T}$, and it is possible to construct only one dimensionless number, $\xi = (z/\sqrt{Dt})$. Therefore, just from dimensional analysis, it can be concluded that the concentration field does not vary independently with z and t, but depends only on the combination $\xi = (z/\sqrt{Dt})$. If this inference is correct, it should be possible to express the conservation equation 3.46 in terms of the variable ξ alone. When z and t are expressed in terms of ξ, the concentration equation becomes

$$-\left(\frac{z}{2D^{1/2}t^{3/2}}\right)\frac{\partial c^*}{\partial \xi} = \frac{D}{Dt}\frac{\partial^2 c^*}{\partial \xi^2} \tag{3.47}$$

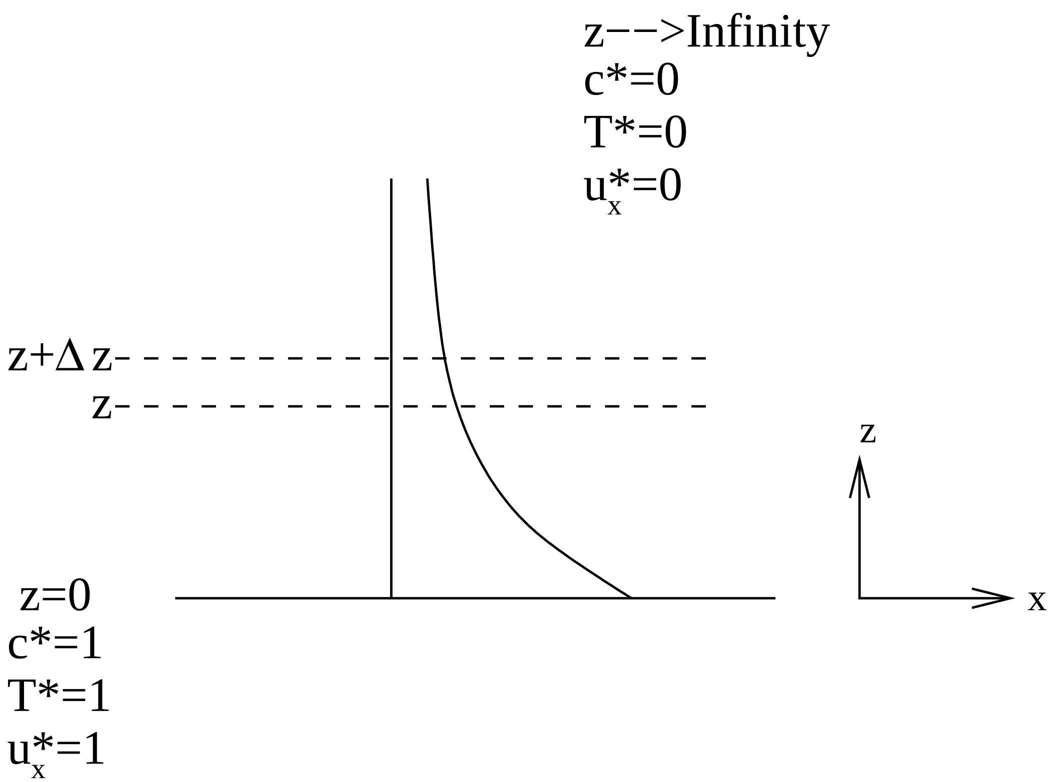


Figure 3.1: Configuration for similarity solution for unidirectional transport.

After multiplying throughout by t, the equation for the concentration field reduces to

$$\frac{\xi}{2}\frac{\partial c^*}{\partial \xi} + \frac{\partial^2 c^*}{\partial \xi^2} = 0 \tag{3.48}$$

Equation 3.48 validates the earlier inference, based on dimensional analysis, that the non-dimensionalised concentration field is only a function of ξ, and contains z, t or D only in the combination $(z/\sqrt{Dt})$.

It is also necessary to transform the boundary and initial conditions, 3.43, 3.44 and 3.45 into conditions for the ξ coordinate. The transformed boundary conditions are

$$c^* = 0 \quad \text{as } z \to \infty \text{ at all } t \to \quad \text{as } \xi \to \infty \tag{3.49}$$
$$c^* = 1 \quad \text{at } z = 0 \text{ at all } t > 0 \to \quad \text{at } \xi = 0 \tag{3.50}$$
$$c^* = 0 \quad \text{at } t = 0 \text{ for all } z > 0 \to \quad \text{as } \xi \to \infty \tag{3.51}$$

Note that the original conservation equation, 3.46, is a second order differential equation in z and a first order differential equation in t, and so this requires two boundary conditions in the z coordinate and one initial condition. The conservation equation expressed in terms of ξ is a second order differential equation, which requires just two boundary conditions for ξ. From equation 3.50 and 3.51, it can be seen that one of the boundary conditions for $z \to \infty$ (equation 3.43) and the initial condition $t = 0$ (equation 3.45) turn out to be identical conditions for $\xi \to \infty$.

Equation 3.48 is a second order ordinary differential equation for $c^*(\xi)$, which can be easily solved to obtain

$$c^*(\xi) = C_1 + C_2 \int_{\xi}^{\infty} d\xi' \exp\left(-\frac{\xi'^2}{4}\right) \tag{3.52}$$

The constants C_1 and C_2 are determined from the conditions $c^* = 1$ at $\xi = 0$, and $c^* = 0$ for $\xi \to \infty$, to obtain

$$c^*(z/\sqrt{Dt}) = \left(1 - \frac{1}{\sqrt{\pi}} \int_0^{(z/\sqrt{Dt})} d\xi' \exp\left(-\frac{\xi'^2}{4}\right)\right) \tag{3.53}$$

The solution 3.53 for $c^*(z/\sqrt{Dt})$ is shown as a function of $(z/\sqrt{Dt})$ in figure 3.2. From this solution, we see that c^* decreases to about 0.48 at $(z/\sqrt{Dt}) = 1.0$, and to about 0.16 at $(z/\sqrt{Dt}) = 2.0$,and further to about 0.034 at $(z/\sqrt{Dt}) = 3.0$. For $(z/\sqrt{Dt}) > 3.0$, the scaled concentration field is

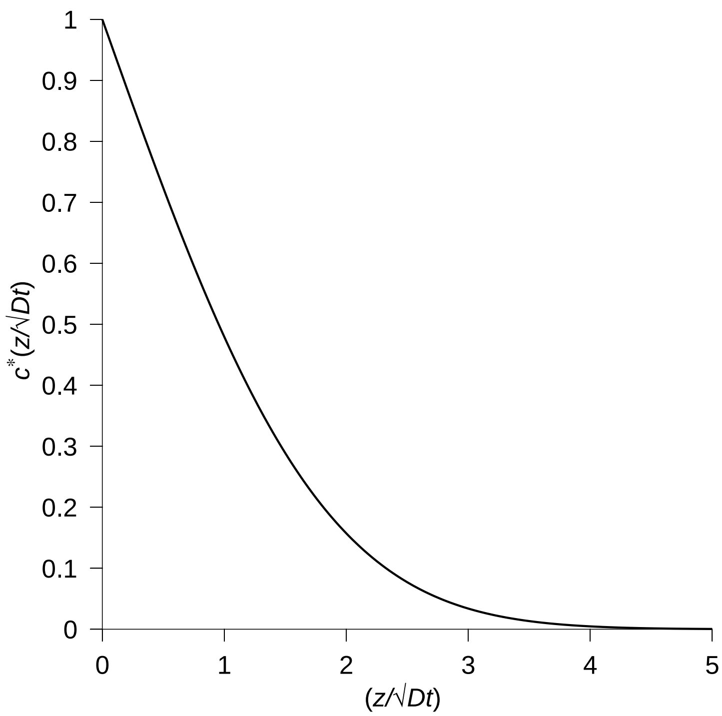


Figure 3.2: The solution equation 3.53 for $c^*(z/\sqrt{Dt})$ as a function of $(z/\sqrt{Dt})$.

close to zero. Therefore, the length scale for the variation of the concentration (penetration depth) is $\sqrt{Dt}$. This length scale is a function of time, and it increases proportional to $t^{1/2}$.

As discussed at the beginning of this section, the solution equation 3.53 is valid only when the penetration depth is small compared to the distance between plates, or $\sqrt{Dt} \ll H$, or $t \ll (H^2/D)$. When the penetration depth becomes comparable to H, a similarity solution cannot be used, because the length scale H is also relevant, and the scaled z co-ordinate can be defined as $z^* = (z/H)$. The similarity reduction here was possible because the boundary conditions 3.49 and 3.50 were applied at $z = 0$ and $z \to \infty$ respectively. Since there is no other length scale, there are three dimensional quantities, z, t and D, and from these it was possible to form only one dimensionless group on the basis of dimensional analysis. However, the similarity solution method is more general, and does not rely on dimensional analysis alone, as shown in the next problem. This method forms the basis of boundary layer theories to be discussed later.

3.2.6 Steady diffusion into a falling film

This problem is a simplification of the actual diffusion in a falling film, which involves a combination of convection and diffusion. We discuss this now, even though it is not an unsteady diffusion problem, because the solution is a similarity solution similar to that for unsteady diffusion into an infinite fluid.

A thin film of fluid flows down a vertical surface with a constant velocity U in the x direction. At the gas-liquid interface, the liquid is in contact with a gas which is soluble in the liquid. The concentration of gas in the liquid at the entrance is c_0, while the concentration of gas at the liquid-gas interface is c_1. The difference in concentration between the initial concentration in the liquid and the concentration at the interface acts as a driving force for diffusion. The z coordinate is perpendicular to the gas-liquid interface, which is located at $z = 0$. As the liquid flows down, the gas is dissolved in the liquid and carried by the fluid in the streamwise x direction, as shown in figure **??**. Therefore, there is a variation in concentration with the z co-ordinate. However, the system is at steady state, and does not vary in time.

The mass conservation equation can be obtained by carrying out a shell balance over a differential volume, as shown in figure **??**. In this case, there is transport due to fluid convection in the streamwise (x) direction, and diffusion due to a concentration gradient in the z direction. There is diffusion in the x direction as well, because the concentration is not a constant in that direction. However, under certain conditions (which we will discuss at the end), the diffusion in this direction is much smaller the the convective transport due to the mean fluid flow.

The terms in the mass balance equation, 3.1, are as follows. Since the system is at steady state, there is no change in the concentration with time, and the term on the left side of equation 3.1 is zero. There is mass entering the differential volume at the right surfaces at z, and mass leaving at the left surface at $z+\Delta z$, and $z+\Delta z$ due to diffusion. These are given by equations 3.3 and 3.4. In addition, there is also mass entering the top surface at x, and leaving the bottom surface at $x + \Delta x$ due to convection,

$$\left(\begin{array}{c} \text{Mass entering} \\ \text{surface at } x \end{array} \right) = Uc|_x \, \Delta y \Delta z \tag{3.54}$$

$$\left(\begin{array}{c} \text{Mass leaving} \\ \text{surface at } x + \Delta x \end{array} \right) = Uc|_{x+\Delta x} \Delta y \Delta z \tag{3.55}$$

Therefore, the mass balance equation is,

$$j_z|_z \Delta x \Delta y - j_z|_{z+\Delta z} \Delta x \Delta y + Uc|_x \Delta y \Delta z - Uc|_{x+\Delta x} \Delta y \Delta z = 0 \tag{3.56}$$

If the above equation is divided by $\Delta x \Delta y \Delta z$, we obtain,

$$\frac{j_z|_z - j_z|_{z+\Delta z}}{\Delta z} + \frac{Uc|_x - Uc|_{x+\Delta x}}{\Delta x} = 0 \tag{3.57}$$

Taking the limit $\Delta x \to 0$ and $\Delta z \to 0$, we obtain,

$$\frac{\partial(Uc)}{\partial x} = -\frac{\partial j_z}{\partial z} \tag{3.58}$$

The Fick's law for the mass flux, $j_z = -D(\partial c/\partial z)$, is substituted into equation 3.57, to obtain,

$$U\frac{\partial c}{\partial x} = D\frac{\partial^2 c}{\partial z^2} \tag{3.59}$$

Here, the term on the left has been simplified because the velocity U is a constant. Using equation 3.34 for the scaled concentration field, the diffusion equation becomes,

$$U\frac{\partial c^*}{\partial x} = D\frac{\partial^2 c^*}{\partial z^2} \tag{3.60}$$

Two boundary conditions are required in the z direction, since equation 3.60 is a second order differential equation in the z, while one 'initial' condition is required in the x direction equation 3.60 is first order in x. In the z direction, the concentration at the liquid gas interface at $z = 0$ is c_1, while the concentration far from the surface in the limit $z \to \infty$ is c_0. Therefore, the boundary condition for the scaled concentration field is,

$$c^* = 1 \ \text{at} z = 0 \tag{3.61}$$

$$c^* = 0 \ \text{for} z \to \infty \tag{3.62}$$

In addition, the concentration at $x \leq 0$ is c_0, for all z, because the liquid has not yet come into contact with the gas. Therefore, the 'initial' condition at $x = 0$ is,

$$c^* = 0 \ \text{at} x = 0 \ \text{for z ¿ 0} \tag{3.63}$$

If we compare equation 3.60 and the unsteady conservation equation 3.46, we see that equation 3.60 can be obtained from equation 3.46 by substituting (x/U) instead of t. The boundary and initial conditions, 3.61 and 3.62, and the 'initial' condition 3.63, can also be obtained from the boundary and initial conditions of the unsteady problem, 3.73, 3.74 and 3.42, if we substitute (x/U) instead of t. Therefore, the solution for equation 3.60 is obtained by substituting (x/U) instead of t in the solution 3.53.

$$c^*(z/\sqrt{Dx/U}) = \left(1 - \frac{1}{\sqrt{\pi}} \int_0^{(z/\sqrt{Dx/U})} d\xi' \exp\left(-\frac{\xi'^2}{4}\right)\right) \tag{3.64}$$

The diffusion in the falling film is an example of a similarity solution where we have not used dimensional analysis. The similarity variable $\xi = (z/\sqrt{Dx/U})$ is not a dimensional necessity, since there are four dimensional variables, x, z, U and D, and only two dimensions, $\mathcal{L}$ and $\mathcal{T}$. However, the similarity between the equations 3.46 for the unsteady diffusion equation and equation 3.60 for the falling film can be used to obtain the solution 3.64. Here, the penetration depth, $\sqrt{Dx/U}$ increases proportional to $\sqrt{x}$, where x is the downstream distance.

We can now examine the assumptions made at the beginning of the calculation. One assumption is that the penetration depth $\sqrt{Dx/U}$ is small compared to the width of the fluid layer, H, or, $(x \ll UH^2/D)$. Since $\mathrm{Pe}_H = (UH/D)$ is a Peclet number based on the fluid velocity and the depth of the flowing layer, this condition requires that $(x/H) \ll Pe_H$. As fluid travels downstream, the distance x becomes comparable to $H\mathrm{Pe}_H$, and the penetration depth is comparable to H. At this point, the similarity solution can no longer be used.

A second assumption is that the velocity U is independent of z over lengths comparable to the penetration depth. In real flows, there is a variation in the velocity near the surface, and the constant velocity approximation is valid only if hte variation in the velocity over a distance comparable to the penetration depth is small compared to the velocity itself. The velocity field close to the surface can be expanded in a Taylor series about its value at the surface,

$$U(z) = U(0) + z \left.\frac{dU}{dz}\right|_{z=0} + \frac{z^2}{2} \left.\frac{d^2U}{dz^2}\right|_{z=0} + \ldots \tag{3.65}$$

At a liquid-gas interface, the shear stress exerted by the gas on the liquid is zero to a good approximation. Due to this, the velocity gradient (dU/dz)

is zero at the surface. Therefore, the variation in velocity near the surface, $U(z) - U(0) = (z^2/2)(d^2U/dz^2)$. Since the penetration depth is $\sqrt{Dx/U}$, the variation in the velocity over distances comparable to the penetration depth is $(Dx/2U)(d^2U/dz^2)$. The velocity near the surface can be considered constant if,

$$\frac{Dx}{U}\frac{d^2U}{dz^2} \ll U \tag{3.66}$$

at the surface, or

$$x \ll \left(\frac{D}{U^2}\frac{d^2U}{dz^2}\right)^{-1} \tag{3.67}$$

Since $(d^2U/dz^2) \sim (U/H^2)$, the above condition reduces to,

$$\frac{x}{H} \ll \frac{UH}{D} \tag{3.68}$$

Therefore, the above condition is also equivalent to $(x/H) \ll \mathrm{Pe}_H$, the condition for the penetration depth to be small compared to the flow depth.

A third assumption is that diffusion along the streamwise direction is small compared to convection. The flux j_x in the downstream direction is $D(\partial c/\partial x) \sim (Dc/x)$. The flux due to the mean velocity Uc. Therefore, the flux due to convection is large compared to that due to diffusion for $(Ux/D) \gg 1$, or $x \gg (D/U)$. This condition can be written as,

$$\frac{x}{H} \gg \mathrm{Pe}_H^{-1} \tag{3.69}$$

From conditions 3.68 and 3.69 it is clear that this analysis can be used only for high Peclet number flows, $\mathrm{Pe}_H \gg 1$, that is, when convective transport is large compared to diffusive transport over a distance comparable to H.

The solution 3.64 can be used to obtain a correlation for the Sherwood number for the flow down an inclined plane. Consider a flow of depth H and length L in the downstream direction. The flux at the surface at a position z is given by,

$$\begin{aligned} j_z|_{x,z=0} &= -D\left.\frac{\partial c}{\partial z}\right|_{z=0} \\ &= -\frac{D(c_1 - c_0)}{\sqrt{Dx/U}}\left.\frac{dc^*}{d\xi}\right|_{\xi=0} \\ &= \frac{1}{\sqrt{\pi}}\sqrt{\frac{U}{Dx}}(c_1 - c_0) \end{aligned} \tag{3.70}$$

The average flux $\bar{j}_z$ over the length L is,

$$\begin{aligned}
\bar{j}_z &= \frac{1}{L}\int_0^L dx\, j_z|_{x,z=0} \\
&= \frac{D(c_1-c_0)}{L}\sqrt{\frac{2U}{D\pi}}\int_0^L dx\, x^{-1/2} \\
&= \frac{2D(c_1-c_0)}{\sqrt{\pi}}\sqrt{\frac{U}{DL}} \\
&= \frac{2D(c_1-c_0)}{L\sqrt{\pi}}\mathrm{Pe}_L^{1/2} \qquad (3.71)
\end{aligned}$$

where $\mathrm{Pe}_L = (UL/D)$ is the Peclet number based on the length L. The Sherwood number is the non-dimensional average flux,

$$\begin{aligned}
\mathrm{Sh} &= \frac{\bar{j}_z L}{D(c_1-c_0)} \\
&= \frac{2}{\sqrt{\pi}}\mathrm{Pe}_L^{1/2} \\
&= 1.12883\mathrm{Pe}_L^{1/2} \qquad (3.72)
\end{aligned}$$

3.2.7 Diffusion in a channel of finite width:

Next we consider the problem of diffusion in a channel bounded by two walls of infinite extent in the $x-y$ plane, separated by a distance H in the z direction, as shown in figure 3.3. Initially, the concentration of the fluid in the channel is equal to c_0. At $t=0$, the concentration of the solute on the wall at $z=0$ is instanteneously increased to c_1, while the concentration on the surface at $z=H$ is equal to c_0. We would like to determikne the variation of the concentration in the z co-ordinate and in time.

The concentration field is first expressed in terms of the scaled concentration field c^* by equation 3.34. The diffusion equation, obtained by a shell balance as before, is given by equation 3.46. However, there is a modification in the boundary conditions,

$$c^* = 0 \qquad \text{at } z = H \text{ at all } t \qquad (3.73)$$

$$c^* = 1 \qquad \text{at } z = 0 \text{ at all } t > 0 \qquad (3.74)$$

$$c^* = 0 \qquad \text{at } t = 0 \text{ for all } z > 0 \qquad (3.75)$$

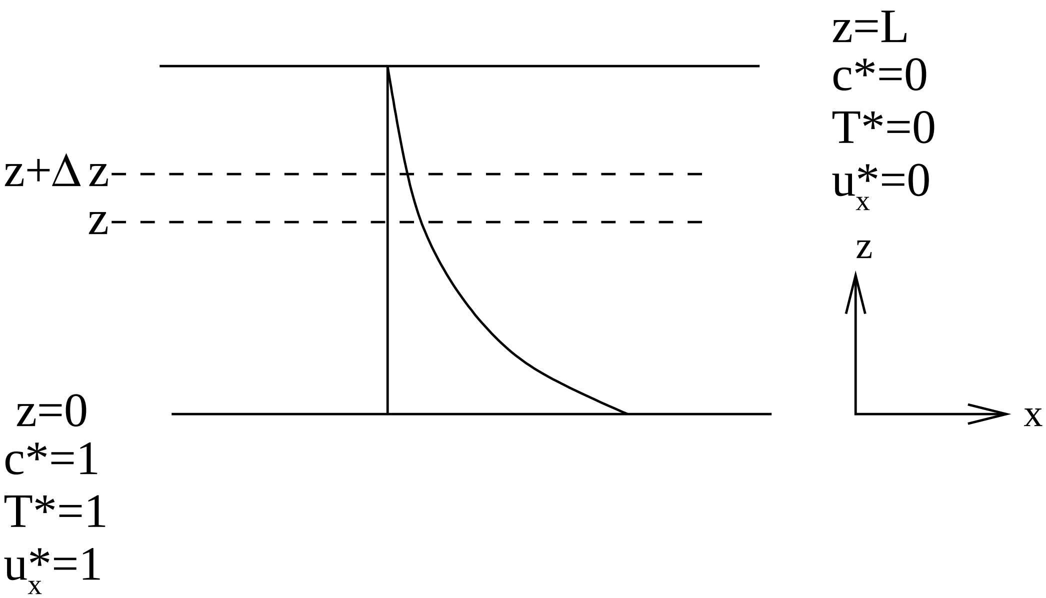


Figure 3.3: Configuration for similarity solution for unidirectional transport.

In this case, it is not possible to reduce the problem using a similarity transform, because there is an additional length scale L in the problem, and so the z coordinate can be scaled by H. A scaled z coordinate is defined as $z^* = (z/L)$, and the diffusion equation in terms of this coordinate is

$$\frac{\partial c^*}{\partial t} = \frac{D}{L^2}\frac{\partial^2 c^*}{\partial z^{*2}} \tag{3.76}$$

The above equation suggests that it is appropriate to define a scaled time coordinate $t^* = (Dt/L^2)$, and the conservation equation in terms of this scaled time coordinate is

$$\frac{\partial c^*}{\partial t^*} = \frac{\partial^2 c^*}{\partial z^{*2}} \tag{3.77}$$

The boundary conditions, in terms of the scaled coordinates z^* and t^*, are

$$c^* = 0 \qquad \text{at } z^* = 1 \text{ at all } t^* \tag{3.78}$$

$$c^* = 1 \qquad \text{at } z^* = 0 \text{ at all } t^* > 0 \tag{3.79}$$

$$c^* = 0 \qquad \text{at } t^* = 0 \text{ for all } z^* > 0 \tag{3.80}$$

We briefly note that similar problems can be framed for heat and momentum transfer. In the heat transfer problem, the two plates and the fluid are initially at temperature T_0. At $t = 0$, the temperature of the bottom plate is instanteneously increased to T_1. In the momentum transfer problem, the fluid and the two plates are initially at rest. At $t = 0$, a constant velocity $u_x = U$ is imparted to the bottom plate. In both cases, non-dimensional concentration, temperature and velocity fields can be defined according to equations 3.34, 3.35 and 3.36. The conservation equations for the scaled temperature and velocity fields are identical to equation 3.77, except that $t^* = (t\alpha/H^2)$ for the heat transfer problem, and $t^* = (t\nu/H^2)$ for the momentum transfer problem. The boundary and initial conditions for T^* and u_x^* are also identical to equations 3.78, 3.79 and 3.80. Therefore, the solutions for the T^* and u_x^* are identical to those obtained here for c^*, except for the definition of the dimensionless time t^* which contains the thermal diffusivity in the heat transfer problem, and the momentum diffusivity in the momentum transfer problem.

The solution method involves separating the concentration field into a steady and transient part. In the long time limit, $t^* \to \infty$, the concentration field will attain a steady state value c_s^* which is independent of time. This steady state concentration field is obtained by solving 3.77 with the time derivative set equal to zero, and the solution for the steady concentration field is a linear concentration profile in equation 3.39,

$$c_s^* = (1 - z^*) \tag{3.81}$$

The concentration can be separated into a steady and a transient part,

$$c^* = c_s^* + c_t^*, \tag{3.82}$$

where c_t^* is the difference between the actual concentration and the concentration at steady state. The reason for this decomposition will become clear a little later.

The conservation equation for the unsteady concentration field is identical to that for the original concentration field, because $(\partial c_s^*/\partial t^*) = 0$ and $(\partial^2 c_s^*/\partial z^{*2}) = 0$,

$$\frac{\partial c_t^*}{\partial t^*} = \frac{\partial^2 c_t^*}{\partial z^{*2}} \tag{3.83}$$

However, the boundary condition for c_t^* is different from that for the c^*, and is obtained by subtracting c_s^* from c^* at the boundaries,

$$c_t^* = 0 \qquad \text{at } z^* = 1 \text{ at all } t^* \tag{3.84}$$

$$c_t^* = 0 \qquad \text{at } z^* = 0 \text{ at all } t^* > 0 \tag{3.85}$$

$$c_t^* = -c_s^* \qquad \text{at } t^* = 0 \text{ for all } z^* > 0 \tag{3.86}$$

Equation 3.83 can be solved by the method of ‘separation of variables’, where the unsteady concentration field is expressed as two functions, one of which is only a function of t^*, while the other is only a function of z^*.

$$c_t^* = \Theta(t^*)Z(z^*) \tag{3.87}$$

This is inserted into the conservation equation 3.83, and the equation is divided by the production ΘZ, to obtain

$$\frac{1}{\Theta}\frac{d\Theta}{dt^*} = \frac{1}{Z}\frac{d^2 Z}{dz^{*2}} \tag{3.88}$$

In equation 3.88, the left side is only a function of t^*, while the right side is only a function of z^*. From this, it can be inferred, as follows, that these two functions have to be constants independent of z^* and t^*. To infer this, assume that these two functions are not constants, and that the left side varies as t^* is varied, and the right side varies as z^* is varied. In this case, if we keep z^* a constant and vary t^*, then the left side of 3.88 varies, while the right side remains a constant, and so the equality is destroyed. The only way for the equality to hold, if the left side is only a function of t^* and the right side is only a function of z^*, is if the two sides are constants.

The solution for Z is first obtained by solving

$$\frac{1}{Z}\frac{d^2 Z}{dz^{*2}} = -\alpha^2 \tag{3.89}$$

where α is a positive constant. The reason for choosing the right side of 3.89 to be negative will become apparent a little later. The solution for this equation is

$$Z = C_1 \sin(\alpha z^*) + C_2 \cos(\alpha z^*) \tag{3.90}$$

where C_1 and C_2 are constants to be determined from the boundary conditions. The boundary condition $c_u^* = 0$ ($Z = 0$) at $z^* = 0$ is satisfied for $C_2 = 0$. The boundary condition $ca_u = 0$ at $z^* = 1$ is satisfied if $\alpha = (n\pi)$, where n is an integer. Therefore, the solution for Z which satisfied the boundary conditions in the z^* coordinate is

$$Z = C_1 \sin(n\pi z^*) \tag{3.91}$$

where n is an integer.

The solution for Θ can now be obtained from the equation

$$\frac{1}{\Theta}\frac{d\Theta}{dt^*} = -\alpha^2 = -n^2\pi^2 \tag{3.92}$$

This equation is solved to obtain

$$\Theta = C_3 \exp\left(-n^2\pi^2 t^*\right) \tag{3.93}$$

The final solution for $c_t^* = \Theta(t^*)Z(z^*)$ is

$$c_t^* = C \exp\left(-n^2\pi^2 t^*\right) \sin\left(n\pi z^*\right) \tag{3.94}$$

The solution 3.94 contains the integer n which is as yet unspecified, and the solution 3.94 satisfies the equation 3.83 for any value of n. The most general solution is one which contains a linear combination of the solution 3.94 for different values of n,

$$c_t^* = \sum_{n=1}^{\infty} C_n \exp\left(-n^2\pi^2 t^*\right) \sin\left(n\pi z^*\right) \tag{3.95}$$

The values of the coefficients C_n have to be determined from the initial condition that has not been used so far,

$$\begin{aligned} c_t^* &= -(1 - z^*) \qquad \text{for } t^* = 0 \\ \sum_{n=1}^{\infty} C_n \sin\left(n\pi z^*\right) &= -(1 - za) \end{aligned} \tag{3.96}$$

The coefficients can be determined because the functions $\sin\left(n\pi z^*\right)$ satisfy 'orthogonality conditions',

$$\begin{aligned} \int_0^1 dz^* \sin\left(n\pi z^*\right) \sin\left(m\pi z^*\right) &= 0 \text{ for } n \neq m \\ &= (1/2) \text{ for } n = m \end{aligned} \tag{3.97}$$

To use this condition, the left and right sides of 3.98 are multiplied by $\sin\left(m\pi z^*\right)$, and integrated over the interval $0 \leq z^* \leq 1$, to obtain

$$\begin{aligned} C_m &= -2\int_0^1 dz^* \sin\left(m\pi z^*\right)(1 - z^*) \\ &= -\frac{2}{m\pi} \quad \text{for odd } n \end{aligned} \tag{3.98}$$

Therefore, the final solution for the concentration field, which includes the steady part ($c_s^* = (1 - z^*)$) and the transient part is,

$$c_z^* = (1 - z^*) - \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin{(n\pi z^*)} \exp{(-n^2\pi^2 t^*)} \tag{3.99}$$

It is now time to examine the reason for choosing the constant on the right side of 3.89 as a negative number. If we had chosen the right side of 3.89 to be *positive*, the solution for Z consists of an exponentially growing and an exponentially decaying function. In this case, it can easily be verified that the boundary condtions $c_t^* = 0$ at $z^* = 0$ and at $z^* = 1$ can be satisfied only if $C_1 = 0$ and $C_2 = 0$. In addition, the solution for Θ in equation 3.93 would have been a function that is exponentially increasing in time, and therefore, there is no steady solution in this case. Since we have chosen the constant to be negative, the unsteady solution c_t^* decays exponentially in time, and goes to zero in the long time limit. Also, the solutions for $Z(z^*)$ are sine functions, which can be chosen to satisfy the conditions $c_t^* = 0$ at $z^* = 0$ and $z^* = 1$.

Further, the separating the concentration field into a steady and a transient part is also now clear. The transient part of the concentration field has homogeneous boundary conditions ($c_t^* = 0$) at both the spatial boundaries, $z^* = 0$ and $z^* = 1$. However, the initial condition ($c_t^* = -c_s^*$ at $t^* = 0$) is inhomogeneous. Therefore, there is no forcing for the transient part of the concentration field at the boundaries, but there is forcing at the initial time $t^* = 0$. The concentration field generated by the forcing at $t^* = 0$ then decays exponentially with time.

The homogeneous spatial boundary conditions $c_t^* = 0$ at $z^* = 0$ and $z^* = 1$ is also essential for another reason. In equation 3.91, we were able to obtain discrete values for the constant $\alpha = n\pi$ only because the solution was a sine function with $Z = 0$ at $z^* = 0$ and 1. This enabled us to get a discrete 'spectrum' of solutions with discrete eignevalues ($n\pi$) and a corresponding set of basis functions $\sin{(n\pi z^*)}$. The solution $Z(z^*)$ was then written as a linear combination of these eigen functions.

The 'orthogonality' conditions 3.97 for determining the constants C_n in equation 3.98 can be physically interpreted as follows. We define the basis functions $S_n = \sin{(n\pi z^*)}$, and we define the inner product

$$\langle S_n, S_m \rangle = \int_0^1 dz^* \sin{(n\pi z^*)} \sin{(m\pi z^*)} \tag{3.100}$$

This inner product is non-zero only when $n = m$, and is zero when $n \neq m$,

$$\langle S_n, S_m \rangle = \frac{\delta_{nm}}{2} \tag{3.101}$$

where 'Kronecker delta' $\delta_{nm} = 1$ for $n = m$, and $\delta_{nm} = 0$ for $n \neq m$. The basis functions S_n are analogous to the basis vectors in a three-dimensional co-ordinate system, while the 'inner product' is analogous to the dot product of unit vectors. The dot product of basis vector with another basis vector is zero, sinc they are perpendicular to each other. In a similar manner, the inner product of two different basis functions is zero, and they are orthogonal to each other. However, there is an infinite number of basis functions for the solution 3.99 for c^*, in contrast to the three unit vectors in three dimensional space.

The concept of inner products can be used to express the orthogonality conditions in a compact manner. In this, the solution 3.95 for c_t^* can be written as,

$$c_t^*(z^*, t^*) = \sum_{n=0}^{\infty} C_n S_n \exp\left(-n^2 \pi^2 t^*\right) \tag{3.102}$$

At $t^* = 0$, the initial condition is $c_t^* = -c_s^* = -(1 - z^*)$. Therefore,

$$\sum_{n=0}^{\infty} C_n S_n = -(1 - z^*) \tag{3.103}$$

We take the inner product of both right and left sides with the basis function S_m.

$$\sum_{n=0}^{\infty} C_n \langle S_m, S_n \rangle = -\langle (1 - z^*), S_m \rangle \tag{3.104}$$

Due to the orthogonality relation 3.101, the above equation reduces to

$$\sum_{n=0}^{\infty} C_n (\delta_{mn}/2) = (C_m/2) = -\langle (1 - z^*), S_m \rangle \tag{3.105}$$

In the summation on the left, δ_{mn} is zero only when $m = n$, and so the entire summation reduces to $(C_m/2)$. Thus, the orthogonality condition 3.105 enables us to determine all the constants in the expansion 3.95, and construct the final solution.

Physically, we are expanding the solution $Z(z^*)$ in a set of basis functions $S_n = \sin(n\pi z^*)$, which consists of a set of sine functions as shown in figure

??. This is a complete basis set, and any function can be expressed as a linear combination of these basis functions. The basis functions are also orthogonal, because the inner product of two different basis functions (equation 3.100) is zero. This orthogonality condition is used to obtain the coefficients in the expansion for $Z(z^*)$. This procedure, of expanding the solutions in a complete and orthogonal basis set, and using the orthogonality relations to determine the terms in the expansions, will be used whenever the separation of variables procedure is carried out.

The solution 3.99 for c^* is a summation over an infinite set of basis functions S_n, and so an exact solution requires the evaluation of an infinite number of terms. However, the n^{th} term in the series is proportional to $\exp(-n^2\pi^2 t^*)$. Therefore, at a fixed time t^*, it is possible to get a good numerical approximation $c^*_{\approx}$ by truncating the series at a finite value of n. The value of n required to obtain the desired accuracy can be estimated as follows. The error $E_{c^*}(p)$, which is the difference between the exact solution c^* and the approximate solution $c^*_{\approx}$ obtained by truncating the series at $n = p - 1$, and neglecting terms for $n \geq p$, is,

$$\begin{aligned} E_{c^*}(p) &= c^* - c^*_{\approx} \\ &= \sum_{n=p}^{\infty} \frac{2}{n\pi} \sin(n\pi z) \exp(-n^2\pi^2 t^*) \end{aligned} \tag{3.106}$$

Since the modulus of $\sin(n\pi z^*)$, the upper bound on the incurred in the approximation is,

$$E_c^{*max}(p) = \sum_{n=p}^{\infty} \frac{2}{n\pi} \exp(-n^2\pi^2 t^*) \tag{3.107}$$

For large values of n, the above summation can be approximated by an integral,

$$\begin{aligned} E_c^{*max}(p) &= \int_p^{\infty} dn \left(\frac{2}{n\pi} \exp(-n^2\pi^2 t^*) \right) \\ &\leq \int_{(p/\pi\sqrt{t^*})}^{\infty} dn' \left(\frac{2}{n'\pi} \exp(-n'^2) \right) \end{aligned} \tag{3.108}$$

The function $E_c^{*max}(p)$ is shown as a function of $(p/\pi\sqrt{t^*})$ in figure 3.4. This upper bound on the error decreases quite rapidly with an increase in $(p/\pi\sqrt{t^*})$. It is less than 0.05 for $p \geq 1.1\pi\sqrt{t^*}$, and it is less than 0.01

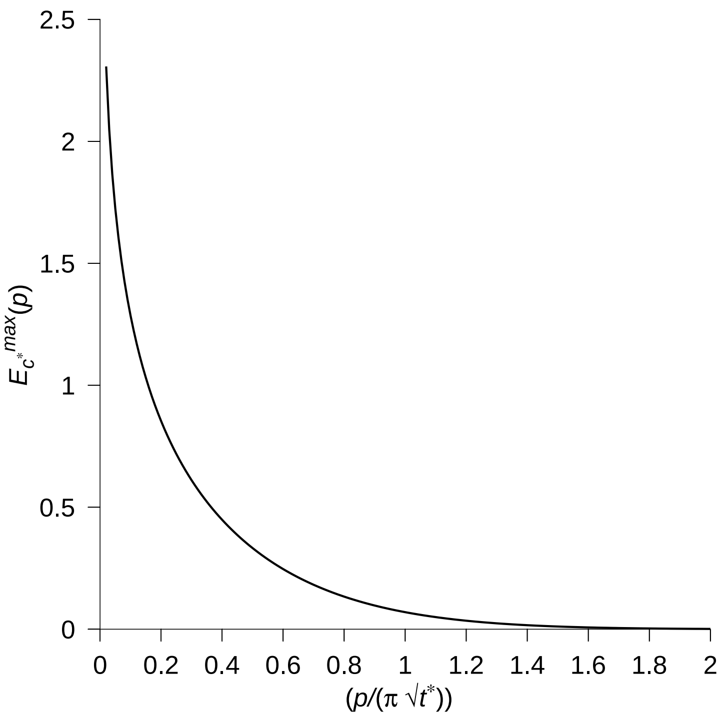


Figure 3.4: The upper bound on the error, $E_c^{*max}(p)$ (equation 3.108), as a function of $(p/\pi\sqrt{t^*})$.

for $p \geq 1.52\pi\sqrt{t^*}$. Thus, good numerical solutions can be obtained with a relatively small number of terms in the series expansion equation 3.99.

The numerical solutions for $c^*(z^*, t^*)$, equation 3.99, are shown as a function of z^* for different values of t^* in figure 3.5. The solid lines show the results when the series is truncated after 20 terms, while the dashed lines show the results when the series is truncated after 5 terms. There is very little difference between the two results for $t^* \geq 0.01$, indicating that it is sufficient to include just five terms in the series even at very short times, $t^* = 0.01$. However, the effect of truncation is clearly observed for $t^* < 0.01$, where the result obtained by truncation after five terms displays an oscillatory behaviour, and it has not yet converged. In contrast, smooth variations are obtained when 20 terms are included in the series. It is also seen that the solution for $t^* = 1$ is indistinguishable from the linear solution for $t^* \to \infty$, and the maximum difference between the solution at $t^* = 1$ and $t^* \to \infty$ is 10^{-4}. This shows that the system has attained steady state, to a very good approximation, at $t^* = 1$.

3.2.8 Oscillatory flow

This example is used to illustrate the use of complex variables in problems where the forcing on the fluid is oscillatory in time. Consider the flow between two flat plates at $z = 0$ and $z = H$, shown in figure 3.6, with the modification that the plate has an oscillatory velocity $U = U\cos(\omega t)$. The differential equation for the velocity field is given by equation 3.29, with $f_x = 0$. As usual, the scaled co-ordinate and velocity are defined as $z^* = (z/H)$ and $u_x^* = (u_x/U)$. However, the scaled time co-ordinate is defined a little differently in the present case. Since there is a time period $(2\pi/\omega)$ associated with the oscillation of the bottom plate, the scaled time can be defined as $t^* = \omega t$. With this, the momentum conservation equation, 3.29, is,

$$\frac{\omega H^2}{\nu}\frac{\partial u_x^*}{\partial t^*} = \frac{\partial^2 u_x^*}{\partial z^{*2}} \tag{3.109}$$

where $\mathrm{Re}_\omega = (\omega H^2/\nu)$ is a Reynolds number based on the frequency of oscillations and the fluid thickness H. The boundary conditions in this case, analogous to 3.38 and 3.38, are,

$$u_x^* = \cos(t^*) \qquad \text{at } z^* = 0 \tag{3.110}$$

$$u_x^* = 0 \qquad \text{at } z^* = 1 \tag{3.111}$$

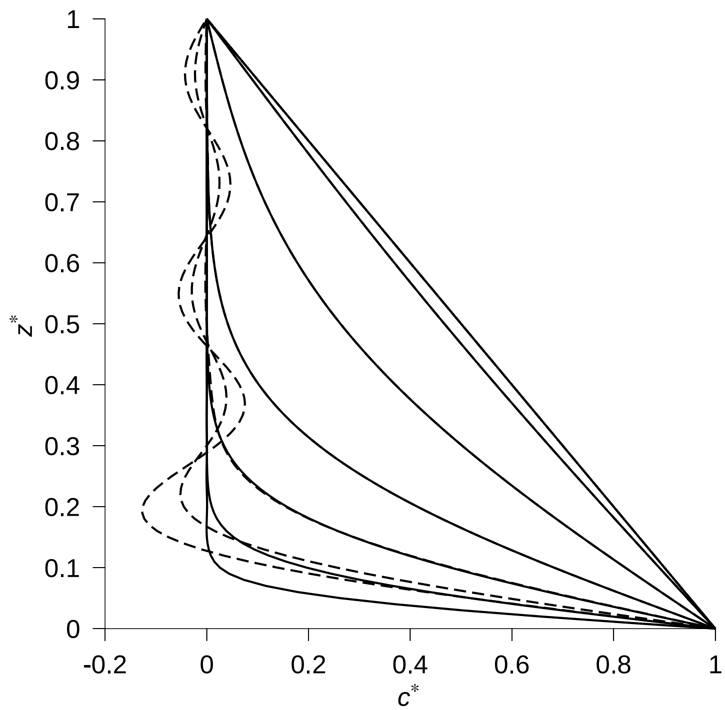


Figure 3.5: Numerical solutions for $c^*(z^*, t^*)$, equation 3.99, as a function for c^* for $t^* = 0.001$, $t^* = 0.003$, $t^* = 0.01$, $t^* = 0.03$, $t^* = 0.1$, $t^* = 0.3$, $t^* = 1.0$ and $t^* \to \infty$.

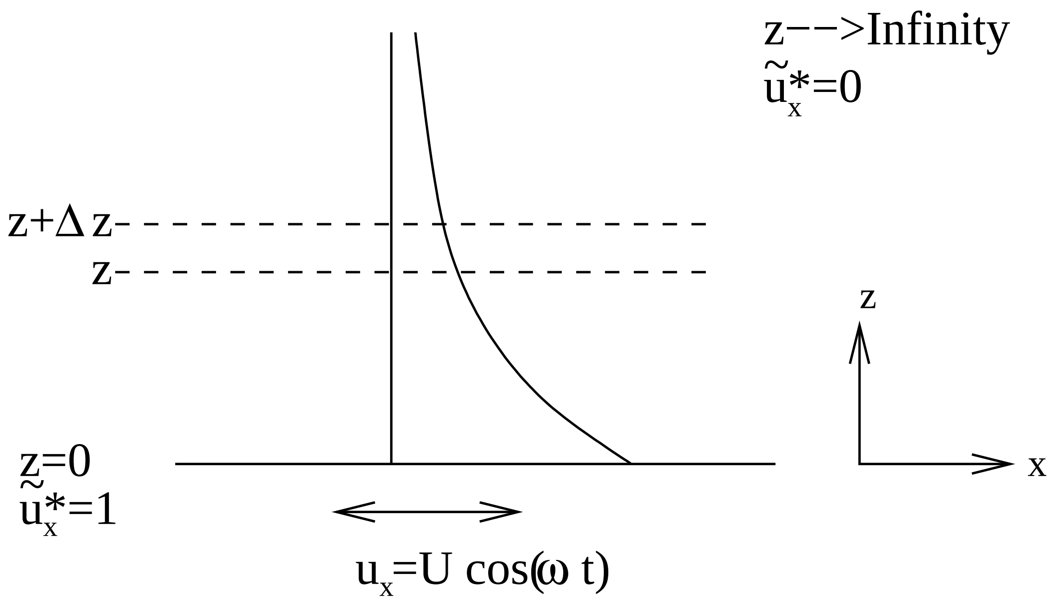


Figure 3.6: Oscillatory flow at a flat surface.

The solution procedure is simplified if the boundary condition is recast as follows. The 'complex' velocity field $u_x^\dagger$ is defined as a velocity field that satisfies the same differential equation as 3.109,

$$\mathrm{Re}_\omega \frac{\partial u_x^\dagger}{\partial t^*} = \frac{\partial^2 u_x^\dagger}{\partial z^{*2}} \tag{3.112}$$

but which satisfies the boundary conditions

$$u_x^\dagger = \exp{(\imath t^*)} \qquad \text{at } z^* = 0 \tag{3.113}$$

$$u_x^\dagger = 0 \qquad \text{at } z^* = H \tag{3.114}$$

where $\imath$ is the square root of -1. The solution to the differential equation 3.109, with the boundary condition 3.110 and 3.111, is the real part of the solution to the differential equation 3.112 with the boundary conditions 3.113 and 3.114. Since dealing with exponential functions is easier than dealing with sines and cosines, it is more convenient to solve equation 3.112 with boundary conditions 3.113 and 3.114, for $u_x^\dagger$, and then take the real part of the solution to obtain the solution u_x^* of equation 3.109.

The differential equation 3.112 for the velocity field is a linear differential equation, since all terms in the equation contain only the first power of

$u_x^\dagger$. This first order differential equation is driven by a wall which is oscillatory wall velocity with frequency ω. When a linear system is driven by wall motion of frequency ω, the response of the system also has the same frequency ω. (This is not true if the system is non-linear, since forcing of a certain frequency will generate response at different harmonics of this base frequency). Therefore, the time dependence of the velocity field in the fluid can be considered to be of the form

$$u_x^\dagger = \tilde{u}_x(z^*) \exp(\imath t^*) \tag{3.115}$$

When this form is inserted into the differential equation 3.112, and divided by $\exp(\imath t^*)$, the resulting equation is an ordinary differential equation for $\tilde{u}_x^*$.

$$\imath \mathrm{Re}_\omega \tilde{u}_x^* = \frac{\partial^2 \tilde{u}_x^*}{\partial z^{*2}} \tag{3.116}$$

The boundary conditions for $u_x^\dagger$ (3.113 and 3.114), when expressed in terms of τ_x, become,

$$\tilde{u}_x^* = 1 \qquad \text{at } z^* = 0 \tag{3.117}$$
$$\tilde{u}_x^* = 0 \qquad \text{at } z^* = 1 \tag{3.118}$$

This equation is easily solved to obtain

$$\tilde{u}_x^*(z^*) = C_1 \exp(\sqrt{\imath \mathrm{Re}_\omega} z^*) + C_2 \exp(-\sqrt{\imath \mathrm{Re}_\omega} z^*) \tag{3.119}$$

The constants C_1 and C_2 are determined from the boundary conditions 3.117 and 3.118,

$$\tilde{u}_x^*(z^*) = \frac{\exp(\sqrt{\imath \mathrm{Re}_\omega} z^*) - \exp(\sqrt{\imath \mathrm{Re}_\omega}(2 - z^*))}{1 - \exp(2\sqrt{\imath \mathrm{Re}_\omega})} \tag{3.120}$$

The physical velocity field, which is the real part of the product of $\tilde{u}_x^*$ and $\exp(\imath t^*)$, is

$$u_x^*(z^*) \quad = \quad \mathrm{Real}\left[\frac{\exp(\sqrt{\imath \mathrm{Re}_\omega} z^*) - \exp(\sqrt{\imath \mathrm{Re}_\omega}(2 - z^*))}{1 - \exp(2\sqrt{\imath \mathrm{Re}_\omega})} \exp(\imath t^*)\right] \tag{3.121}$$

The numerical solutions for the velocity u_x^* are shown as a function of z^* in figure **??** for $\mathrm{Re}_\omega = 0.1, 1.0, 10.0$ and 100.0. It is clear that the velocity

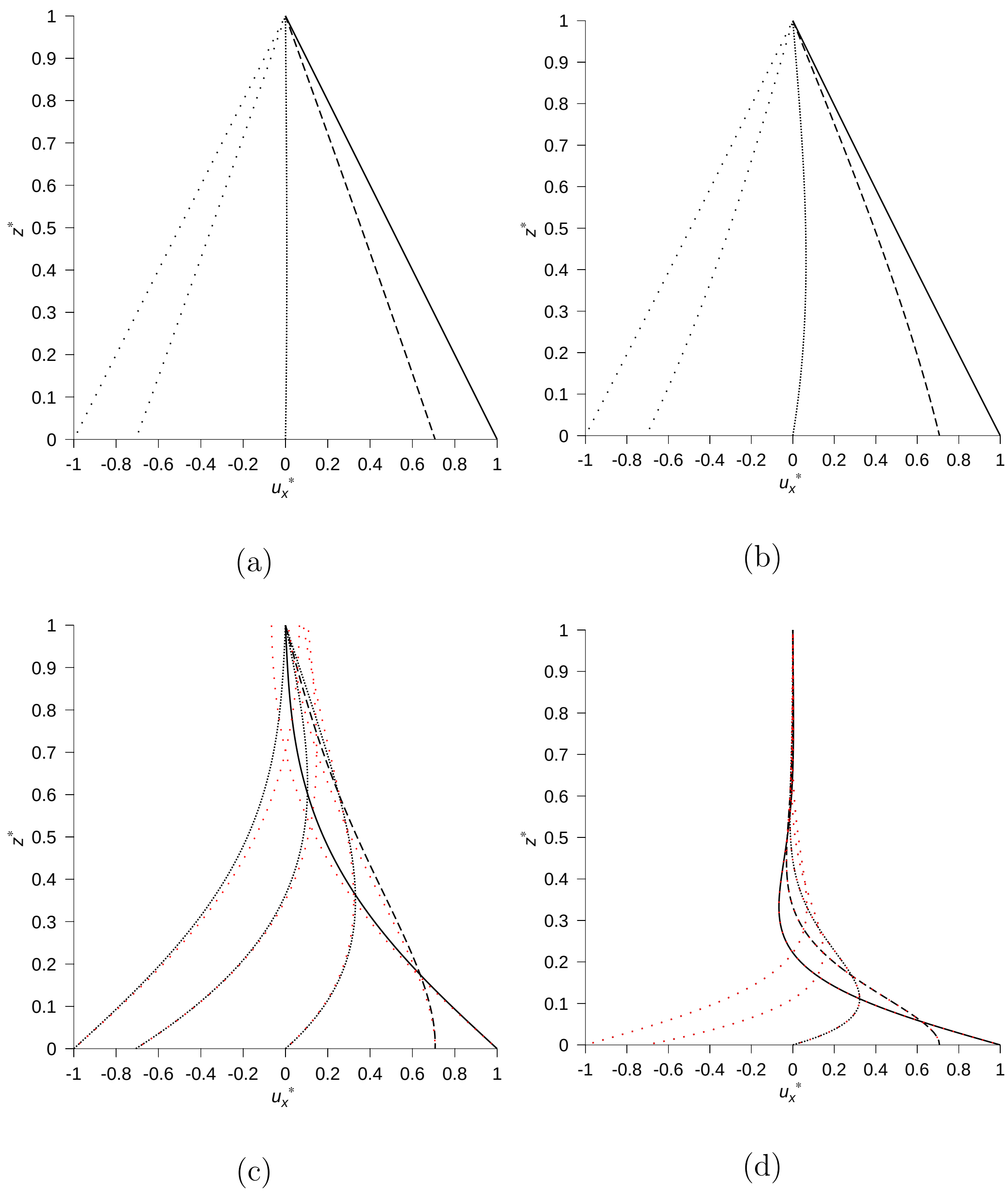


Figure 3.7: The velocity u_x^*, equation 3.121, as a function of z^*, for $\mathrm{Re}_\omega = 0.1$ (a), $\mathrm{Re}_\omega = 1.0$ (b), $\mathrm{Re}_\omega = 10.0$ (c) and $\mathrm{Re}_\omega = 100.0$ (d). The profiles, form right to left, are at $t^* = 0$, $t^* = (\pi/4)$, $t^* = (\pi/2)$, $t^* = (3\pi/4)$ and $t^* = \pi$.

profiles are nearly linear functions of z^* for $\mathrm{Re}_\omega = 0.1$. In contrast, the fluid motion is confined to a thin layer near the moving plate for $\mathrm{Re}_\omega = 100.0$, and there is almost no motion in the bulk of the fluid. The physical reason for this is as follows.

In the limit $\mathrm{Re}_\omega \to 0$, the solution for the fluid velocity, 3.120, is

$$u_x^*(z^*) = (1 - z^*)\cos(t^*) \tag{3.122}$$

The physical reason for this result is as follows. The Reynolds number Re_ω can be interpreted as the product of the frequency ω, and the time required for the momentum to diffuse across the length of the channel, (H^2/ν). For $\mathrm{Re}_\omega \ll 1$, the time period $(2\pi/\omega)$ of variation of the velocity of the bottom plate is long compared to the time required for momentum to diffuse across the channel. In this case, the velocity profile at any instant is the linear velocity profile for a steady flow, in which the velocity at $z^* = 0$ is $(u_x^* = \cos(t^*)$, the instanteneous velocity of the bottom plate at that instant. Therefore, we recover the linear profile for the steady flow between two plates, but with the velocity amplitude varying in time proportional to $\cos(t^*)$.

In the limit $\omega^* \gg 1$, the fluid velocity field is given by

$$\begin{aligned} u_x^*(z^*) &= \mathrm{Real}(\exp(-\sqrt{\imath \mathrm{Re}_\omega} z^*)\exp(\imath t^*)) \\ &= \exp(-\sqrt{\mathrm{Re}_\omega/2} z^*)(\cos(\sqrt{\mathrm{Re}_\omega/2} z^*)\cos(t^*) - \sin(\sqrt{\mathrm{Re}_\omega/2} z^*)\sin(t^*)) \end{aligned} \tag{3.123}$$

In this case, the velocity field decreases over a distance $z^* \sim (1/\sqrt{\mathrm{Re}_\omega/2})$ from the surface. This is because the frequency of oscillation is large compared to the time required for diffusion of momentum across the channel, and the momentum diffuses only to a distance comparable to $(H/\sqrt{\mathrm{Re}_\omega})$. Beyond this distance, the momentum generated during the positive and negative parts of a cycle cancel out, and the fluid velocity approaches zero. Thus, the 'penetration depth' of the fluid velocity field is proportional to $(H/\sqrt{\mathrm{Re}_\omega})$ in the limit $\mathrm{Re}_\omega \gg 1$.

3.3 Effect of bulk flow and reaction in mass transfer

In this section, the special effects of bulk flow and reactions on the solutions for unidirection mass transfer problems are examined.

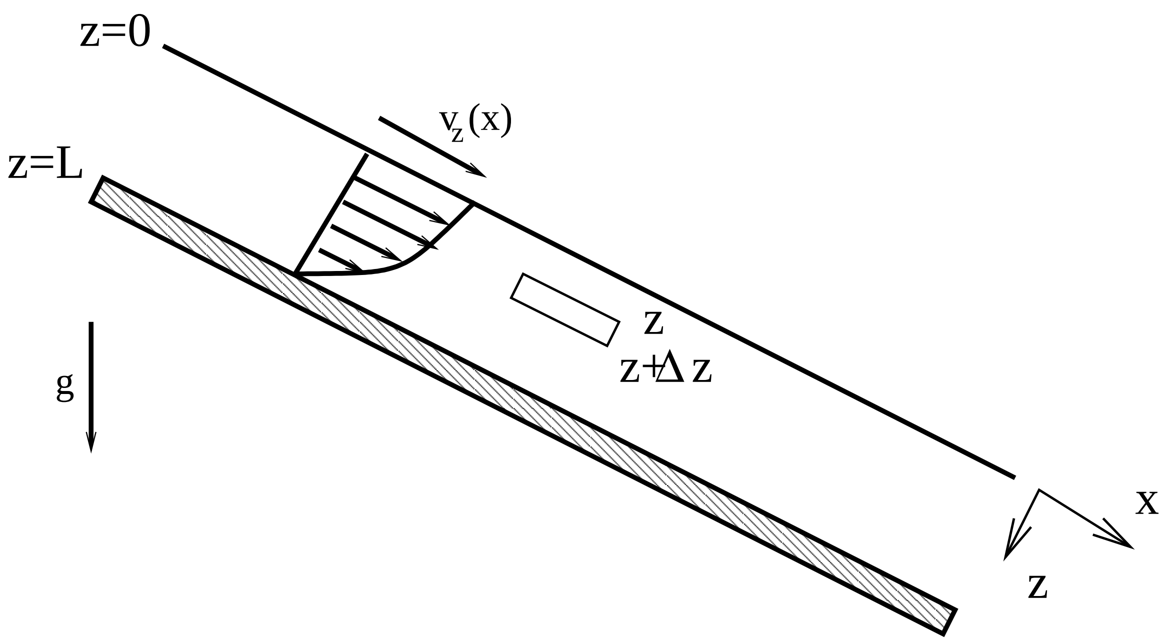


Figure 3.8: Flow down an inclined plane.

Flow down an inclined plane:

The flow of a fluid film along an inclined plane, as shown in figure 3.8, is an example of flow due to the gravitational force acting on a fluid. The plane is inclined at an angle θ to the horizontal. The fluid layer of thickness H is flowing in the x direction along the inclined plane. The film is assumed to be of infinite extent in the y direction, and there is no variation of the velocity in this direction. The flow is fully developed, so that there is no variation of the velocity in the x direction, and we consider both a steady flow and an unsteady flow. The only non - zero component of the velocity, u_x, is a function of the coordinate z, and could be a function of time as well.

The momentum conservation equation 3.29 for the velocity field is of the form,

$$\rho \frac{\partial u_x}{\partial t} = \mu \frac{\partial^2 u_x}{\partial z^2} + \rho g \sin(\beta) \tag{3.124}$$

where $\rho g \sin(\theta)$ is the component of the gravitational force acting in the x direction. The boundary conditions for the flow are as follows. At the bottom surface $z = 0$, the fluid is in contact with a stationary surface, and so the fluid velocity is zero at this surface.

$$u_x = 0 \quad \text{at} \quad z = 0 \tag{3.125}$$

At the top surface $z = H$, the liquid is in contact with a gas. Since the gas viscosity is small compared to the viscosity of the liquid, the shear stress exerted by the gas on the liquid is small. Therefore, we can use the 'zero shear stress' condition at the top surface,

$$\tau_{xz} = \mu \frac{\partial u_x}{\partial z} = 0 \quad \text{at} \quad z = H \tag{3.126}$$

Thus, the gradient of the velocity in the z direction is zero at the free surface.

The scaled z co-ordinate in equation 3.124 can be defined as $z^* = (z/H)$, as before. How do we define a scaled velocity u_x^*, since there is no prescribed velocity at the boundaries? The scaling for the velocity can be determined from the momentum conservation equation **??** itself, since this equation contains a 'source' of momentum due to the gravitational force. If we substitute $z = z^* H$, and divide the entire equation by $\rho g \sin(\theta)$, we obtain,

$$\frac{\partial u_x^*}{\partial t^*} = \frac{\partial^2 u_x^*}{\partial z^{*2}} + 1 \tag{3.127}$$

where the scaled velocity $u_x^* = (\mu u_x/(H^2 \rho g \sin(\theta)))$, and $t^* = (t\nu/H^2)$ is the scaled time. Equation 3.127 is a linear partial differential equation for u_x^*, which contains an *inhomogeneous term*, 1, due to the body force. In contrast, the boundary conditions for the scaled velocity u_x^* are both homogeneous,

$$u_x^* = 0 \quad \text{at} \quad z^* = 0 \tag{3.128}$$

$$\frac{\partial u_x^*}{\partial z^*} = 0 \quad \text{at} \quad z^* = 1 \tag{3.129}$$

Therefore, in the present problem, there is no forcing at the boundaries, and the flow is driven by forcing within the flow itself due to the gravitational flow. In contrast, in the flow between two flat plates in section **??**, there is no forcing within the flow, and the flow is driven by the motion of the boundary.

At steady state, the time derivative in equation 3.127 is set equal to zero,

$$\frac{\partial^2 u_x^*}{\partial z^{*2}} + 1 = 0 \tag{3.130}$$

This solution of this equation, which satisfies the boundary conditions 3.128 and 3.129, is,

$$u_x^* = z^* - \frac{z^{*2}}{2} \tag{3.131}$$

The dimensional velocity can be easily determined from the above,

$$u_x = \frac{\rho g \sin(\theta)}{\mu}\left(zH - \frac{z^2}{2}\right) \tag{3.132}$$

Various average quantities can be determined once this velocity profile is known.

1. the maximum velocity, u_{xm}, is clearly at $x = 0$

$$u_{xm} = \frac{\rho g H^2 \sin(\theta)}{2\mu} \tag{3.133}$$

2. The total flow rate is determined from

$$\begin{aligned} Q &= \int_0^H dz \int_0^W dy u_x \\ &= \frac{\rho g W H^3 \sin(\theta)}{3\mu} \end{aligned} \tag{3.134}$$

3. The mean velocity can be calculated from

$$\begin{aligned} \bar{u}_x &= \frac{Q}{h} \\ &= \frac{\rho g W H^2 \sin(\theta)}{3\mu} \end{aligned} \tag{3.135}$$

4. The film thickness δ can be expressed in terms of the flow rate as

$$H = \left(\frac{3\mu Q}{\rho g W \sin(\theta)}\right)^{1/3} \tag{3.136}$$

5. The total force on the inclined surface in the z direction is given by

$$F = \int_0^L dz \int_0^W dy\ \tau_{xz}|_{x=h} \tag{3.137}$$

$$= \rho g h L W \sin(\theta) \tag{3.138}$$

This is just equal to the weight of the fluid in the z direction under steady flow conditions.

Next, we consider the start-up of the flow in an initially stationary film of fluid down an inclined plane. The film is initially horizontal, and at $t = 0$, the film is inclined at an angle θ with respect to the horizontal. As before, there is no variation of the velocity in the streamwise x direction, but there is a flow development in time. The momentum conservation equation is 3.127, with boundary conditions 3.128 and 3.129. In addition, at the initial time $t^* = 0$, the fluid is stationary, and so the initial condition is,

$$u_x^* = 0 \quad \text{at} \quad t^* = 0 \quad \text{for all } z^* > 0 \tag{3.139}$$

The unsteady problem is solved by the method of separation of variables. First, the velocity is separated into two parts, the steady part u_{xs}^* and the transient part u_{xt}^*,

$$u_x^* = u_{xs}^* + u_{xt}^* \tag{3.140}$$

The equation for the steady velocity field, 3.130, is subtracted from the equation for the total velocity field, 3.127, to obtain the equation for the transient part of the velocity field.

$$\frac{\partial u_{xt}^*}{\partial t} = \frac{\partial^2 u_x^*}{\partial z^{*2}} \tag{3.141}$$

The boundary conditions for the total velocity and the steady velocity field, 3.128 and 3.129, are both homogeneous, the boundary conditions for the transient velocity field are also homogeneous,

$$u_{xt}^* = 0 \quad \text{at} \quad z^* = 0 \tag{3.142}$$

$$\frac{\partial u_{xt}^*}{\partial z^*} = 0 \quad \text{at} \quad z^* = 1 \tag{3.143}$$

Finally, the initial condition for u_{xt}^* is obtained by subtracting the steady solution u_{xs}^* (equation 3.131) from the initial condition 3.139,

$$u_{xt}^* = -(z^* - (z^{*2}/2)) \quad \text{at} \quad t^* = 0 \quad \text{for all } z^* \tag{3.144}$$

As in the case of the transient flow in a channel in section **??**, the transient part of the velocity field has homogeneous boundary conditions, but the initial condition is inhomogeneous.

The separation of variables procedure provides the following solution for the transient part of the velocity field,

$$u_{xt}^* = \sum_{n=1}^{\infty} (C_n \sin{(\alpha_n z^*)} + D_n \sin{(\alpha_n z^*)}) \exp{(-\alpha_n^2 t^*)} \tag{3.145}$$

where the coefficients C_n, D_n and the eigenvalues α_n are chosen so that the boundary conditions 3.142 and 3.143, and the initial condition 3.144, are satisfied. The boundary condition 3.142 is satisfied only if $D_n = 0$, for all n, while the boundary condition 3.143 is satisfied if $\alpha_n = (2n+1)\pi/2$ for $n = 1, 2, \ldots$. Therefore, the solution for u^*_{xt} which satisfies the boundary conditions at $z^* = 0$ and $z^* = 1$ is,

$$u^*_{xt} = \sum_{n=1}^{\infty} C_n \sin\left((2n+1)\pi z^*/2\right) \exp\left(-((2n+1)/2)^2 \pi^2 t^*\right) \tag{3.146}$$

The coefficients C_n determined from the initial condition at $t^* = 0$, equation 3.144.

$$\begin{aligned} u^*_{xt}(z^*, t^* = 0) &= -(z^* - z^{*2}/2) \\ &= \sum_{n=1}^{\infty} C_n \sin\left((2n+1)\pi z^*/2\right) \end{aligned} \tag{3.147}$$

In the solution 3.147, the functions $\sin\left((2n+1)\pi z^*/2\right)$ form a set of basis functions which are orthogonal to each other. The orthogonality relation is,

$$\int_0^1 dz^* \sin\left((2n+1)\pi z^*/2\right) \sin\left((2m+1)\pi z^*/2\right) = \frac{\delta_{mn}}{2} \tag{3.148}$$

This orthogonality relation can be used to determine the coefficients C_n in equation 3.147,

$$\begin{aligned} &\int_0^1 dz^* (-(z^* - z^{*2}/2)) \sin\left((2m+1)\pi z^*/2\right) \\ &= \int_0^1 dz^* \sin\left((2m+1)\pi z^*/2\right) \sum_{n=1}^{\infty} C_n \sin\left((2n+1)\pi z^*/2\right) \\ &= C_m/2 \end{aligned} \tag{3.149}$$

Thus, the coefficients C_m are,

$$C_m = -\frac{1}{\pi^3((2n+1)/2)^3} \tag{3.150}$$

This, the final solution for the unsteady velocity field is,

$$u^*_x = (z^* - z^{*2}/2) - \sum_{n=1}^{\infty} (\pi^3((2n+1)/2)^3)^{-1} \sin\left((2n+1)\pi z^*/2\right) \exp\left(-((2n+1)/2)^2 \pi^2 t^*\right) \tag{3.151}$$

The above analytical results are valid only if the flow is laminar and the streamlines are smooth, so that the flow can be considered steady. These conditions are satisfied for the slow viscous flow of a thin film. As the velocity increases or the film thickness increases, it has been found that there is a transition from a laminar flow ith straight streamlines to a laminar flow with rippling and then to a turbulent flow. The conditions under which these transitions occur is determined by the 'Reynolds number', $Re = (4\rho h\bar{u}_x/\mu)$. A laminar flow without rippling is observed for $Re < 10$, while there is rippling for $10 < Re < 1000$. The flow becomes turbulent when the Reynolds number increases beyond about 1000.

3.3.1 Viscous heating in a channel:

There is a dissipation of energy during the shear flow of a viscous liquid due to fluid friction, and this energy increases the temperature of the fluid. We consider the specific example of a pressure-driven flow in a channel between two infinite flat plates located at $z = 0$ and $z = H$. The temperature at both the bounding surfaces is T_0, but there is an increase in the temperature within the channel due to the heat generated by viscous dissipation. We would like to find out the temperature within the channel. At steady state, the velocity profile in the channel is given by,

$$\begin{aligned} u_x &= -\frac{1}{2\mu}\frac{dp}{dx}z(H-z) \\ &= 4U\left(\frac{z}{H} - \left(\frac{z}{H}\right)^2\right) \end{aligned} \tag{3.152}$$

where (dp/dx) is the pressure gradient, and U is the maximum velocity at the center of the channel.

The rate of dissipation of energy due to fluid friction will be calculated later when we derive the mass, momentum and energy balance equations for a fluid. For a laminar shear flow where the velocity is in the x direction and the velocity variation is in the z direction, the rate of dissipation of energy (per unit volume per unit time), S_e in equation 3.15, is given by,

$$\begin{aligned} S_e &= \tau_{xy}\frac{du_x}{dz} \\ &= \mu\left(\frac{du_x}{dz}\right)^2 \end{aligned} \tag{3.153}$$

where τ_{xy} is the shear stress, and (du_x/dy) is the strain rate. Using this velocity profile in equation 3.152, we find that the dissipation rate per unit volume, S_e, is,

$$S_e = 16U^2 \left(\frac{z}{H} - \left(\frac{z}{H} \right)^2 \right)^2 \tag{3.154}$$

At steady state, the energy balance equation, 3.19, reduces to,

$$k\frac{d^2T}{dz^2} + \frac{16\mu U^2}{H^2} \left(1 - \frac{2z}{h} \right)^2 = 0 \tag{3.155}$$

The boundary conditions are,

$$T = T_0 \text{ at } z = 0 \tag{3.156}$$

$$T = T_0 \text{ at } z = H \tag{3.157}$$

It is natural to define a scaled z co-ordinate, $z^* = (z/H)$, and a scaled temperature, $T^* = ((T - T_0)/T_0)$. Defined this way, the scaled temperature is the ratio of the local temperature rise due to viscous heating and the wall temperature. With this non-dimensionalisation, the energy balance equation becomes,

$$\frac{d^2T^*}{dz^{*2}} + 16\text{Br}(1 - 2z^*)^2 = 0 \tag{3.158}$$

with boundary conditions,

$$T^* = 0 \text{ at } z^* = 0 \tag{3.159}$$

$$T^* = 0 \text{ at } z^* = 1 \tag{3.160}$$

where the Brinkman number is,

$$\text{Br} = \frac{\mu U^2}{kT_0} \tag{3.161}$$

Equation 3.158 can be easily solved, subject to boundary conditions 3.159 and 3.160, to obtain,

$$T^* = \text{Br} \left(\frac{8z^*(1 - z^*)(1 - 2z^* + 2z^{*2})}{3} \right) \tag{3.162}$$

The profile of the scaled temperature, divided by Br, is shown as a function of the scaled z co-ordinate in figure 3.3.1. The temperature profile is very

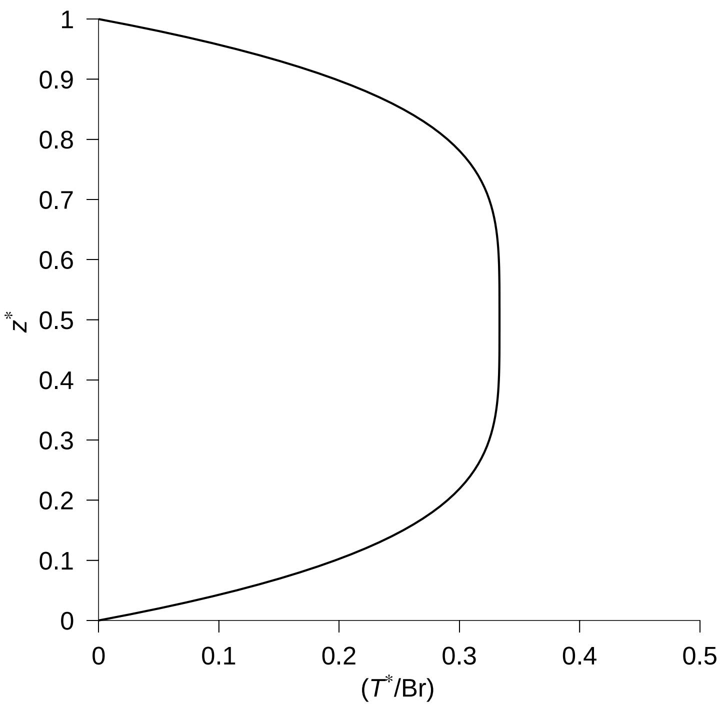


Figure 3.9: The ratio (T^*/Br) of the scaled temperature $T^* = ((T - T_0)/T_0)$, as a function of z^* a channel.

flat at the center of the channel, because the strain rate (du_x/dz) decreases to zero at the center, and the rate of generation also decreases to zero. The rate of generation of heat is a maximum near the wall, where the strain rate is a maximum.

From equation 3.162 for the temperature profile, the fractional increase in the temperature within the channel is given by the Brinkman number. For Br $\ll 1$, the temperature rise in the channel is small compared to the wall temperature, and so change in temperature due to viscous heating can be neglected. Viscous heating also results in a flux of energy across the wall of the channel, which is given by,

$$\begin{aligned} q_z &= -k\frac{dT}{dz} \\ &= -\frac{kT_0}{H}\frac{dT^*}{dz^*} \\ &= \frac{8kT_0(1-2z^*)^3 \mathrm{Br}}{3H} \\ &= \frac{8\mu U^2(1-2z^*)^3}{3H} \end{aligned} \tag{3.163}$$

The heat flux is negative at the bottom surface at $z^* = 0$, because heat is transferred downwards from the fluid to the wall. At $z^* = 1$, the heat flux is positive because heat is transferred upwards to the wall. In both cases, the magnitude of the heat flux is given by $(8\mu U^2/3H)$.

Recall that in dimensional analysis of the heat transfer in a heat exchanger in chapter 1, we had assumed that there is no conversion of mechanical energy to heat energy. The present calculation shows that this assumption is valid only when the flux q_z due to viscous heating, $(8\mu U^2/3H)$, is small compared to the flux due to the temperature difference across the wall of the heat exchanger. When the heat flux due to viscous heating is comparable to that due to the temperature difference across the wall of the tube, it is necessary to include the viscous heating in the energy balance equation, and the Nusselt number will be a function of the Brinkman number as well.

3.3.2 Diffusion with homogeneous reaction

A gaseous reactant A dissolves in a liquid B, and undergoes a first order reaction $A + B \rightarrow AB$, in a tank of height L, as shown in figure 3.10. The mass balance equation, **??** has to be modified in this case due to the presence of a 'consumption' term due to chemical reaction. The mass balance equation at steady state takes the form

$$j_{Az}|_z - j_{Az}|_{z+\Delta z} - kc_A S\Delta z = 0 \tag{3.164}$$

This can be reduced to a differential equation by dividing throughout by $S\Delta z$, and taking the limit $\Delta z \to 0$,

$$\frac{dj_{Az}}{dz} - kC_A = 0 \tag{3.165}$$

If the concentration of A is small, the flux of A is given by

$$j_{Az} = -D_{AB}\frac{dc_A}{dz} \tag{3.166}$$

Inserting this into the concentration equation 3.165, we get

$$-D_{AB}\frac{d^2c_A}{dz^2} + kC_A = 0 \tag{3.167}$$

The boundary conditions are

$$\begin{aligned} c_A = c_{A0} \quad &\text{at} \quad z = 0 \\ j_{Az} = 0 \quad &\text{at} \quad z = H \end{aligned} \tag{3.168}$$

The solution that satisfied both these conditions is

$$\frac{C_A}{C_{A0}} = \frac{\cosh\left((kL^2/D_{AB})^{1/2}(1-(z/L))\right)}{\cosh\left(kL^2/D_{AB}\right)^{1/2}} \tag{3.169}$$

3.3.3 Diffusion in a stagnant film

Water evaporates from a container through a stagnant air film through a glass tube into dry air flowing at the top of the tube, as shown in figure 3.11. If the mole fraction at the surface of the liquid surface is the saturation mole fraction x_{Ws}, and the dry air flowing past the tube does not contain any water, what is the concentration profile of water in the glass tube?

Though the air in the tube is stationary, the mean velocity across any horizontal surface in the tube is not zero, because of the flow of water vapour

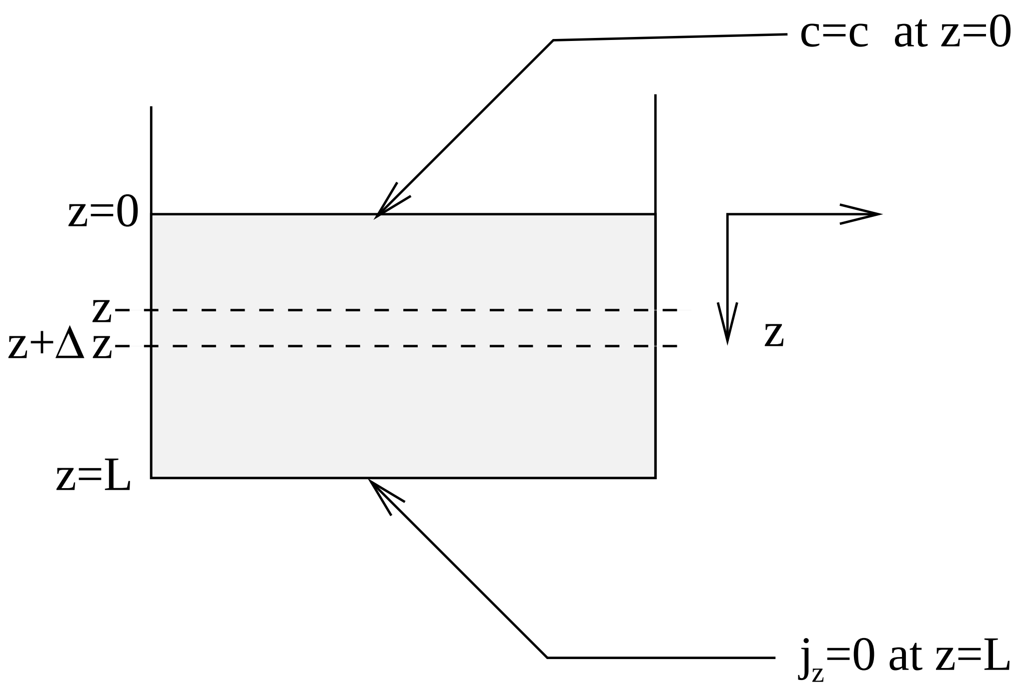


Figure 3.10: Diffison with homogeneous chemical reaction.

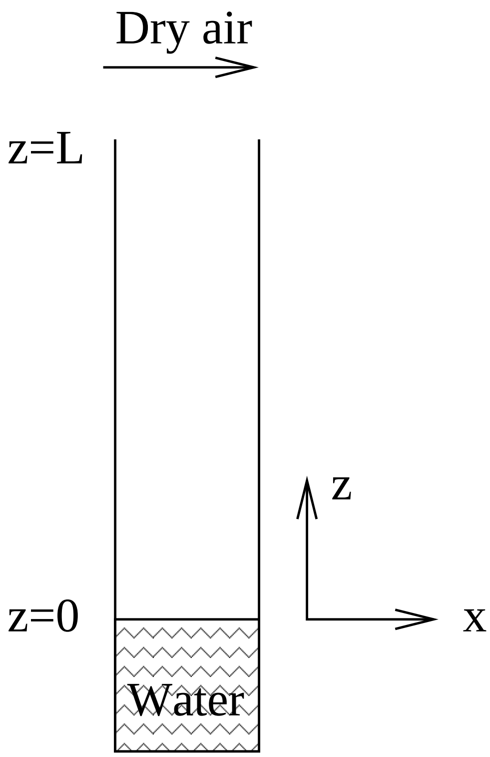


Figure 3.11: Diffusion with bulk flow.

across the surface. The flux of water across a surface, j_{Wz}, contains a component due to the bulk flow, as well as a component due to the diffusion of water across the surface.

$$j_{Wz} = -cD_{WA}\frac{dx_W}{dz} + x_W(j_{Wz} + j_{Az}) \tag{3.170}$$

The last term on the right side of equation 3.170 is the flux of water due to the bulk flow, where the total molar flow rate is the sum of the fluxes of water (j_{Wz}) and air (j_{Az}). In this particular case, the flux of air is identically zero, and so the flux of water vapour across a surface is given by

$$j_{Wz} = -\frac{c}{1 - x_W}\frac{dx_W}{dz} \tag{3.171}$$

The mass balance equation is obtained by writing a flux balance across a section of thickness Δz of the tube, which at steady state provides,

$$j_{Wz}|_{z+\Delta z} - j_{Wz}|_z = 0 \tag{3.172}$$

If the above equation is divided by Δz the differential equation for the flux in the limit $\Delta z \to 0$ is

$$\frac{dj_{Wz}}{dz} = \frac{d}{dz}\left(\frac{1}{1 - x_W}\frac{dx_W}{dz}\right) = 0 \tag{3.173}$$

This equation is solved to obtain

$$-\log(1 - x_W) = A_1 z + A_2 \tag{3.174}$$

The constants A_1 and A_2 are determined from the boundary conditions $x_W = x_{Ws}$ at $z = 0$ and $x = 0$ at $z = l$,

$$\frac{(1 - x_W)}{(1 - x_{Ws})} = \left(\frac{(1 - x_{Wf})}{(1 - x_{Ws})}\right)^{z/l} \tag{3.175}$$

3.4 Cylindrical co-ordinates:

3.4.1 Balance laws:

In the previous section, we had used a Cartesian co-ordinate system to analyse the transport between two flat plates. The Cartesian co-ordinate system was convenient for this geometry, because the boundaries were surfaces on which one of the co-ordinates (the z co-ordinate) is a constant. Due to this, the boundary conditions were applied at a constant value of z. In the case of systems with cylindrical geometry, such as the flow through a pipe, heat conduction across the surface of a tube, or the mass transfer in a cylindrical pore on a catalyst surface, the use of a Cartesian co-ordinate system is complicated, because none of the co-ordinates is a constant on the cylindrical surface. It is more convenient to use a cylindrical co-ordinate system, as shown in figure 3.12. This co-ordinate system has cylindrical symmetry about an 'axis', which is the z axis in figure 3.12. The co-ordinates in the cylindrical co-ordinate system are (r, θ, z), where r is the distance of a point from the z axis, and θ is the angle between the position vector and the x co-ordinate. The third co-ordinate, z, is identical to that in a Cartesian co-ordinate system. In our analysis of unidirectional transport, we will assume that there is variation of concentration, temperature or velocity only in the r direction and in time, and there is no dependence on θ and z.

First, we derive a heat balance equation for the temperature variation in a cylindrical shell of thickness Δr and height Δz at radius r. The terms in the balance equation 3.11 for the energy in the cylindrical shell are as follows.

$$\left(\begin{array}{c}\text{Accumulation of energy}\\ \text{in the shell}\end{array}\right) = \rho C_p (T(x,y,z,t+\Delta t) - T(x,y,x,t)) 2\pi r \Delta r \Delta z \tag{3.176}$$

The total energy entering the shell at r is the product of the heat flux, the surface area, and the time interval Δt,

$$\left(\begin{array}{c}\text{Input of}\\ \text{energy into shell}\end{array}\right) = -\left.(q_r(2\pi r \Delta z \Delta t))\right|_r \tag{3.177}$$

where q_r is the heat flux in the radial direction. Similarly, the total energy leaving the shell at $r + \Delta r$ is

$$\left(\begin{array}{c}\text{Output of}\\ \text{energy from shell}\end{array}\right) = \left.(q_r(2\pi r \Delta z \Delta t))\right|_{r+\Delta r} \tag{3.178}$$

The source of energy in the differential volume is,

$$\left(\begin{array}{c}\text{Source of}\\ \text{energy from shell}\end{array}\right) = S_e(2\pi r \Delta r \Delta z \Delta t) \tag{3.179}$$

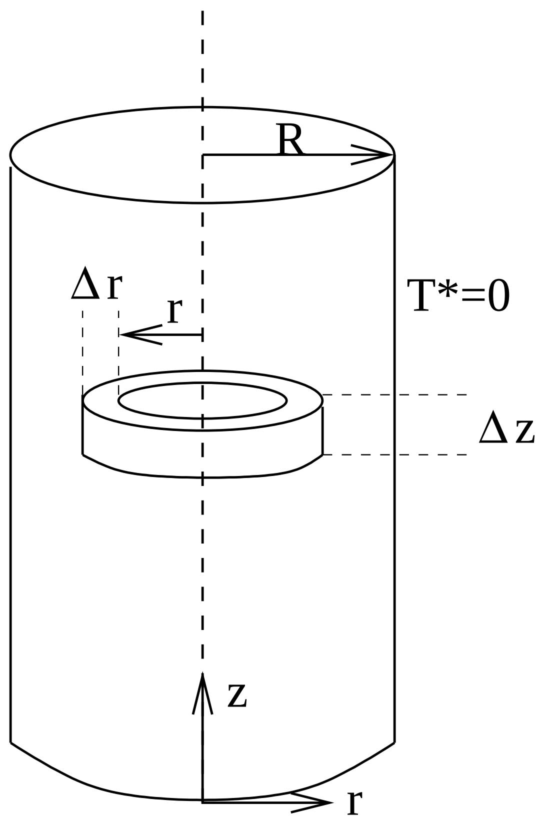


Figure 3.12: Heat diffusion into a cylinder.

where S_e is the amount of energy generated per unit volume per unit time. When these are inserted into the conservation equation 3.11, and divided by $2\pi r\Delta r\Delta z\Delta t$, the net energy balance for the shell is

$$\rho C_p \frac{(T(x,y,z,t+\Delta t) - T(x,y,x,t))}{\Delta t} = \frac{1}{r\Delta r}\left((rq_r)|_r - (rq_r)|_{r+\Delta r}\right) + S_e \tag{3.180}$$

Taking the limit $\Delta r \to 0$ and $\Delta t \to 0$, the partial differential equation for the temperature field is

$$\rho C_p \frac{\partial T}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}(rq_r) + S_e \tag{3.181}$$

The heat flux q_r is related to the temperature gradient in the radial direction by the Fourier's law for heat conduction,

$$q_r = -k\frac{\partial T}{\partial r} \tag{3.182}$$

With this, the energy balance equation becomes,

$$\rho C_p \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) + S_e \tag{3.183}$$

When the thermal conductivity is independent of position, the energy balance equation reduces to,

$$\frac{\partial T}{\partial t} = \alpha\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + (S_e/\rho C_p) \tag{3.184}$$

where $\alpha = (k/\rho C_p)$ is the thermal diffusivity.

It is important to note that there is a variation in the surface area of the shell as r varies. This leads to a more complicated form for the diffusion term in 3.184, in comparison to the second derivative with respect to z in the diffusion from a flat plane, 3.20.

Similar to equation 3.184, the mass conservation equation for a cylindrical co-ordinate system, analogous to equation 3.10 for a Cartesian co-ordinate system, is,

$$\frac{\partial c}{\partial t} = D\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right) + S \tag{3.185}$$

where D is the mass diffusivity and S is the rate of increase of mass per unit volume per unit time.

In section **??**, the momentum conservation equation was written for the velocity u_x, which is a constant in the x direction, but is a function of the direction z between the two plates. In a cylindrical co-ordinate system, there are two velocity components perpendicular to r, the θ co-ordinate u_θ and the z co-ordinate u_z. Momentum balance equations could be written in either of these two directions. A momentum balance equation in the z direction is solved when there is a flow along the axis of the cylindrical co-ordinate system, such as the flow in a pipe. The velocity u_θ is non-zero when there is a flow around the axis of the cylindrical co-ordinate system, with circular streamlines. The momentum balance for u_θ is similar to the energy and mass balance equations 3.184 and 3.185,

$$\rho \frac{\partial u_\theta}{\partial t} = \mu \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\theta}{\partial r} \right) + f_\theta \tag{3.186}$$

where f_θ is the force per unit volume acting in the θ direction.

3.4.2 Heat transfer across the wall of a pipe:

Consider a cylindrical pipe with inner radius R_i and outer radius R_o, as shown in figure **??**. The inner surface is at temperature T_i, while the outer surface is at temperature T_o. We would like to determine the heat flux across the wall of the pipe, which has thermal conductivity k at steady state.

The scaled temperature and distance are defined as $T^* = (T - T_i)/(T_o - T_i)$ and $r^* = (r/R_i)$. The heat balance equation at steady state is,

$$\frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial T^*}{\partial r^*} \right) = 0 \tag{3.187}$$

and the boundary conditions are,

$$T^* = 0 \quad \text{at} \quad r^* = 1 \tag{3.188}$$

$$T^* = 1 \quad \text{at} \quad r^* = (R_o/R_i) \tag{3.189}$$

This can be easily solved to obtain,

$$T^* = \frac{\log(r^*)}{\log(R_o/R_i)} \tag{3.190}$$

Thus, we obtain a logarithmic variation in the temperature along the radius of the tube. The heat flux, q_r, is given by,

$$\begin{aligned} q_r &= -k\frac{\partial T}{\partial r} \\ &= -\frac{k(T_o - T_i)}{R_i}\frac{\partial T^*}{\partial r^*} \\ &= -\frac{k(T_o - T_i)}{r^* R_i \log{(R_o/R_i)}} \end{aligned} \tag{3.191}$$

The total heat transfer rate across the tube wall (per unit length of the tube) is obtained by integrating the heat flux over the surface. At which surface should this be calculated, the inner surface, the outer surface or somewhere in between? The answer is that it does not matter which surface we use, the heat transfer rate is independent of the radius of the surface. This is because in the conservation equation 3.187, there is no generation or consumption of heat in the tube wall, and the temperature is at steady state. Therefore, the heat transferred per unit time is the same across any cylindrical surface in the wall of the tube. If we consider a cylindrical surface of radius r and length L, the heat transferred across this surface per unit time, Q, is $(2\pi r L)$ times the heat flux,

$$\begin{aligned} Q &= (2\pi r L) \times \left(-\frac{k(T_o - T_i)}{r^* R_i \log{(R_o/R_i)}}\right) \\ &= \frac{2\pi k(T_o - T_i)L}{\log{(R_o/R_i)}} \end{aligned} \tag{3.192}$$

It is convenient to define an 'average' radial heat flux based on the thickness of the tube wall, $(R_o - R_i)$,

$$\bar{q}_r = \frac{Q}{2\pi L(R_o - R_i)} \tag{3.193}$$

Based on this definition, the average heat flux from equation 3.192 is,

$$\bar{q}_r = -\frac{k(T_o - T_i)}{(R_o - R_i)\log{(R_o/R_i)}} \tag{3.194}$$

3.4.3 Heat conduction from a wire:

In our discussion of heat conduction into an infinite medium in a Cartesian co-ordinate system, a similarity solution was used because there are no length and time scales in the problem. Such a solution cannot be used for the conduction from a cylinder into an infinite medium, because the radius of the cylinder also provides a length scale which can be used to non-dimensionalise distances. However, there is one special situation where the similarity solution procedure can be used, which is the heat conduction from a thin wire. This problem is of relevance in real applications, because resistance heating is a common method for heating fluids. The problem is as follows.

A resistance heating appratus for a fluid consists of a thin wire immersed in a fluid. In order to design the appratus, it is necessary to determine the temperature in the fluid as a function of he heat flux from the wire. For the purposes of the calculation, the wire can be considered of infinite length so that the heat conduction problem is effectively a two dimensional problem. In addition, the thickness of the wire is considered small compared to any other length scales in the problem, so that the wire is a line source of heat. The wire and the fluid are initially at a temperature T_0. At time $t = 0$, the current is switched on so that the wire acts as a source of heat, and the heat transmitted per unit length of the wire is Q. The thermal conductivity of the liquid is K.

There is an important difference between the above problem and the problem of conduction into an infintite medium in Cartesian co-ordinates. In the latter, the temperature at the surface was instanteneously increased to from T_0 to T_1 at time $t = 0$. In the above problem, we have specified the heat produced per unit length of the wire, Q. This is because the thickness of the wire is small compared to any other length scales, and we take the limit where the radius of the wire goes to zero. (As indicated above, if the radius of the wire is finite, it is possible to scale all lengths by the radius of the wire, and a similarity solution cannot be used). As the thickness goes to zero, the surface area of the wire goes to zero, and so the heat flux (ratio of heat generated and surface area) will go to infinity. An infinite heat flux implies an infinite temperature gradient, and so it is not possible to specify the temperature itself at the surface of the wire. Rather than using a boundary condition at the surface, we will use the condition that the total heat generated per unit length is Q in order to determine the constants in the solution.

The scaled temperature is defined as $T^* = ((T - T_\infty)/T_\infty)$, so that the scaled temperature is zero in the limit $r \to \infty$. The heat conduction equation in the fluid is,

$$\frac{\partial T^*}{\partial t} = \alpha \left(\frac{1}{r} \frac{d}{dr} r \frac{dT^*}{dr} \right) \tag{3.195}$$

One of the boundary conditions are that $T^* \to 0$ in the limit $r \to \infty$. The other boundary condition is a flux condition. The total heat transmitted, per unit length, of the wire is Q. Therefore, the flux from a cylindrical surface of radius r is $Q/(2\pi r)$. Therefore, the requirement at the wire surface, in the limit $r \to 0$, is that

$$-KT_\infty \frac{dT^*}{dr} = \frac{Q}{2\pi r} \tag{3.196}$$

where Q is a constant.

To solve for the temperature field, note that there is no length scale in the problem (the wire is infinitesimal in thickness, and the boundaries are at infinity). Therefore, a similarity solution can be used with the similarity variable $\xi = (r/\sqrt{\alpha t})$. The heat conduction equation, in terms of this variable, is

$$\frac{d^2 T^*}{d\xi^2} + \left(\frac{1}{\xi} + \frac{\xi}{2} \right) \frac{dT^*}{d\xi} = 0 \tag{3.197}$$

This equation can be solved to obtain the

$$\frac{dT^*}{d\xi} = \frac{C}{\xi} \exp\left(-\xi^2/4\right) \tag{3.198}$$

The temperature can be obtained by integrating the above equation with respect to ξ, and realising that $T = 0$ as $\xi \to \infty$.

$$T^* = \int_\infty^\xi d\xi' \frac{C}{\xi'} \exp\left(-\xi'^2/4\right) \tag{3.199}$$

The above equation shows that the temperature at the surface of the wire is undefined, because the integral in equation 3.199 increases proportional to $\log(\xi)$ as $\xi \to \infty$. This was anticipated when we framed the problem, where we defined the total heat generated per unit length of the wire, and not the temperature at the wire itself.

The constant C can be determined from the flux condition, in the limit $r \to 0 (\xi \to 0)$,

$$K \frac{dT^*}{dr} = -\frac{Q}{2\pi r} \tag{3.200}$$

When expressed in terms of ξ, this is equivalent to

$$\frac{dT^*}{d\xi} = -\frac{Q}{2\pi K\xi} \tag{3.201}$$

Substituting solution 3.198 into the above equation, we find that the constant $C = (Q/2\pi K)$. Therefore, the solution for the temperature field is,

$$T^* = \frac{Q}{2\pi k}\int_{\infty}^{\xi} d\xi' \frac{1}{\xi'} \exp\left(-\xi'^2/4\right) \tag{3.202}$$

The solution $(T^*/(Q/2\pi k))$ is shown as a function of $(r/\sqrt{\alpha t})$ in figure **??**.

3.4.4 Heat conduction into a cylinder:

A cylindrical container of radius R containing fluid with temperature T_1 is dipped into a bath at temperature T_0. Assume that the bath is large enough that the conduction of heat into the fluid does not appreciably change the temperature of the bath, and that there is no resistance to heat transfer in the walls of the container. The temperature in the cylinder is to be determined as a function of time and position is to be determined.

The conservation equation 3.184 can be scaled as follows. The scaled radius and time are defined as $r^* = (r/R)$ and $t^* = (tR^2/\alpha)$, since R is the length scale in the radial direction. The scaled temperature is defined as, $T^* = (T - T_0)/(T_1 - T_0)$. The conservation equation 3.184, expressed in these coordinates, is

$$\frac{\partial T^*}{\partial t^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial T^*}{\partial r^*}\right) \tag{3.203}$$

since there is no heat generation within the cylinder.

The boundary condition at the surface of the cylinder, $r^* = 1$, is,

$$T^* = 0 \text{at } r^* = 1 \text{ for all } t^* \tag{3.204}$$

At the center of the cylinder $r^* = 0$, we specify a 'symmetry condition', that the temperature gradient, $(\partial T^*/\partial r^*) = 0$. The reason for this is as follows. Consider a section of the cylinder through a plane passing through the axis, as shown in figure **??** Since the temperature field is axisymmetric, the temperature is the same at equal distances on the right and left of the axis. If the derivative $(\partial T^*/\partial r^*)$ is not zero, then the slope of the temperature profile at the axis from the left will be the opposite of the slope from the right, and so the slope at the center will not be uniquely defined. There is a discontinuity in the slope when the axis is approached from left and right. The slope will be continuous only if $(\partial T^*/\partial r^*) = 0$ at $r^* = 0$. Therefore, symmetry requires that

$$\frac{\partial T^*}{\partial r^*} = 0 \text{at } r^* = 0 \text{ for all } t^* (3.205)$$

At time $t^* = 0$, the temperature is T_0 throughout the cylinder except at the cylinder wall, and so the initial condition is,

$$T^* = 1 \quad \text{at } t^* = 0 \text{ for all } r^* < 1 \tag{3.206}$$

The spatial boundary conditions at $r^* = 1$ and $r^* = 0$, equations 3.204 and 3.205, both homogeneous, whereas the initial condition 3.206 is inhomogeneous. As we had discussed in relation to the separation of variables in Cartesian co-ordinates, the separation of variables procedure provides a discrete set of eigenvalues and basis functions only if we have homogeneous boundary conditions in the spatial co-ordinates. Equations 3.204 and 3.205 provide the required homogeneous boundary conditions in the present problem.

Equation **??** is solved using the method of separation of variables. The final steady state solution is one in which the temperature is uniform, $T^* = 0$, throughout the cylinder. The unsteady solution is solved using the substitution

$$T^* = \mathcal{R}(r^*)\mathcal{T}(t^*) \tag{3.207}$$

where $\mathcal{R}$ is a function of the radial coordinate, and $\mathcal{T}$ is a function of time. The form 3.207 is inserted into the temperature equation, 3.203, and the resulting equation is divided by $\mathcal{R}(r^*)\mathcal{T}(t^*)$ to obtain

$$\frac{1}{\mathcal{T}}\frac{\partial \mathcal{T}}{\partial t^*} = \frac{1}{\mathcal{R}}\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial \mathcal{R}}{\partial r^*}\right) \tag{3.208}$$

The left side of the above equation is only a function of time, while the right side is only a function of r^*. Therefore, the equality can only be satisfied if both sides are equal to constants. From our earlier experience of the separation of variables in Cartesian co-ordinates, we know that this constant has to be negative, so that the function $\mathcal{T}(t^*)$ decreases to zero in the limit $t^* \to \infty$. The right side of the above equation is first solved,

$$\frac{1}{\mathcal{R}}\left(\frac{d^2\mathcal{R}}{dr^{*2}} + \frac{1}{r^*}\frac{d\mathcal{R}}{dr^*}\right) = -\beta^2 \tag{3.209}$$

The solution for equation 3.199 can be obtained by recasting the equation as

$$r^{*2}\frac{d^2\mathcal{R}}{dr^{*2}} + r^*\frac{d\mathcal{R}}{dr^*} + \beta^2 r^*\mathcal{R} = 0 \tag{3.210}$$

The two linearly independent solutions of the above equation 3.210, are special functions called 'Bessel functions', The Bessel functions $J_n(x)$ and $Y_n(x)$, are linearly independent solutions of the equation,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \tag{3.211}$$

In equation 3.210, if we substitute $x = \beta r^*$, we find that the equation reduces to,

$$x^2\frac{d^2\mathcal{R}}{dx^2} + x\frac{d\mathcal{R}}{dx} + x^2\mathcal{R} = 0 \tag{3.212}$$

The solutions to this equation are 'zeroeth order' Bessel functions, with $n = 0$,

$$\mathcal{R} = C_1 J_0(\alpha r^*) + C_2 Y_0(\beta r^*) \tag{3.213}$$

There is no analytical solutions for these functions, and it is necessary to evaluate these numerically. The Bessel functions $J_0(x)$ and $Y_0(x)$ are shown as a function of x in figure 3.4.4. The Bessel functions have an oscillatory dependence on x, but the maximum amplitude of the oscillations decreases and the period increases as x increases. The Bessel function $J_0(x)$ has a maximum value of 1 at $x = 0$. The Bessel function $Y_0(x)$ goes to $-\infty$ for $x \to 0$.

The values of the constants C_1 and C_2 are determined from the boundary conditions at $r^* = 0$ and $r^* = 1$. From equation 3.205, $(d\mathcal{R}/dr^*) = 0$ at $r^* = 0$. From figure 3.4.4, it is observed that the derivative of $J_0(x)$ is zero, whereas the radial derivative of $Y_0(x)$ approaches infinity for $r^* \to 0$. Therefore, the boundary condition at $r^* = 0$ can be satisfied only if $C_2 = 0$.

The second boundary condition, $T^* = 0$ at $r^* = 1$, is used to determine the value of β in equation **??**,

$$J_0(\beta) = 0 \tag{3.214}$$

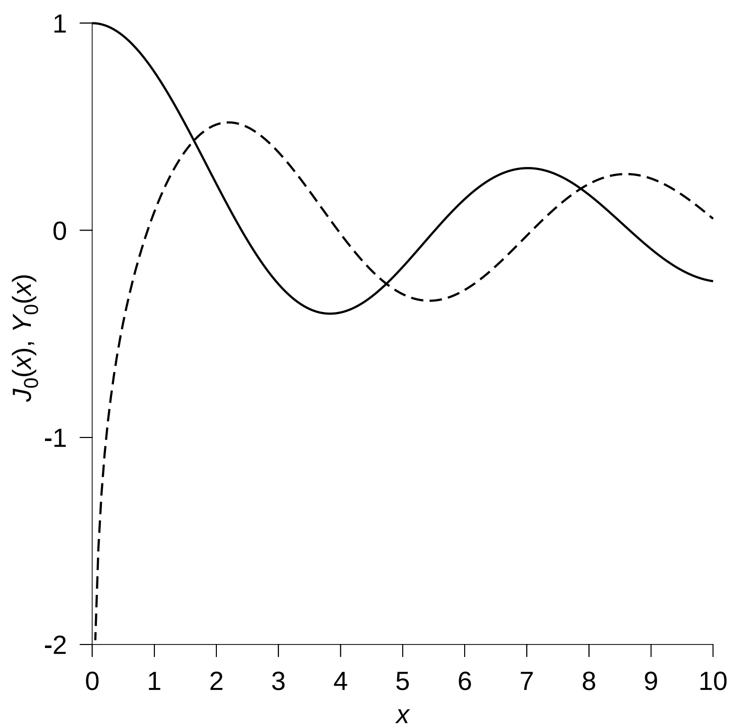


Figure 3.13: The Bessel functions, $J_0(x)$ and $Y_0(x)$ as a function of x.

Since $J_0(\beta)$ is an oscillatory function of β, there are multiple solutions for equation 3.214, which are the points where $J_0(x)$ crosses zero in figure 3.4.4. The first few solutions are, $\beta_1 = 2.40483$, $\beta_2 = 5.52008$, $\beta_3 = 8.65373$, $\beta_4 = 11.79150$ and $\beta_5 = 14.93090$. Thus, the requirement that $\mathcal{R}(r^*) = 0$ at $r^* = 1$ results in a discrete set values for the 'eigenvalue' β_n. This is analogous to the discrete values of $\beta_n = (n\pi)$ in the unsteady heat conduction between two parallel plates in section **??**.

The equation for $\mathcal{T}$,

$$\frac{1}{\mathcal{T}}\frac{d\mathcal{T}}{dt^*} = -\beta_n^2 \tag{3.215}$$

can be solved to obtain

$$\mathcal{T} = \exp\left(-\alpha_n^2 t^*\right) \tag{3.216}$$

With this, the solution for the temperature field is

$$T^* = \sum_{n=1}^{\infty} A_n J_0(\beta_n r^*) \exp\left(-\beta_n^2 t^*\right) \tag{3.217}$$

The coefficients A_n in equation 3.217 are chosen to satisfy the initial condition at $t^* = 0$,

$$\sum_{n=1}^{\infty} A_n J_0(\beta_n r^*) = 1 \tag{3.218}$$

The values of the coefficients can be determined using orthogonality conditions for the Bessel functions, which in this case are

$$\begin{aligned} \int_0^1 r^* dr^* J_0(\beta_n r^*) J_0(\beta_m r^*) &= \frac{1}{2}(J_1(\beta_n))^2 \text{ for } m = n \\ &= 0 \text{ for } m \neq n \end{aligned} \tag{3.219}$$

In order to determine the coefficients, the right and left sides of 3.199 are multiplied by $r^* J_0(\beta_m r^*)$, and integrated from $r^* = 0$ to $r^* = 1$, to obtain

$$\begin{aligned} \frac{1}{2}(J_1(\beta_n))^2 A_n &= \int_0^1 r^* dr^* J_0(\beta_n r^*) \\ &= \frac{J_1(\beta_n)}{\beta_n} \end{aligned} \tag{3.220}$$

This provides the solution for A_n,

$$A_n = \frac{2}{\beta_n J_1(\beta_n)} \tag{3.221}$$

This, the final solution for the temperature field is,

$$T^* = \sum_{n=0}^{\infty} \frac{2}{\beta_n J_1(\beta_n)} J_0(\beta_n r^*) \exp\left(-\beta_n^2 t^*\right) \tag{3.222}$$

When we had solved the unsteady temperature profile in between two flat plates in section **??**, we had expressed the solution in terms of 'basis functions' $\Psi_n = \sin(n\pi z^*)$. In the present problems, the basis functions are,

$$\Psi_n = J_0(\beta_n r^*) \tag{3.223}$$

where β_n are the solutions of equation 3.214. The inner product, equivalent of equation 3.100, is,

$$\langle \Psi_n, \Psi_m \rangle \;\; = \;\; \int x \, dx \, J_0(\beta_n x) J_0(\beta_m x) \tag{3.224}$$

The orthogonality relation, equivalent to equation 3.101, is,

$$\langle \Psi_n, \Psi_m \rangle \;\; = \;\; \frac{J_1(\beta_m)\delta_{mn}}{2} \tag{3.225}$$

In the equivalent mass transfer problem, a cylinder of radius with uniform concentration $c = c_1$ is immersed into a fluid with concentration $c = c_0$ at time $t = 0$, and it is necessary to find the variation of concentration with time in the cylinder. The scaled concentration field is defined as $c^* = (c - c_0)/(c_1 - c_0)$. The concentration equation is identical to 3.222, with c^* substituted for T^*, and $t^* = (tD/R^2)$, where D is the mass diffusion coefficient.

In the equivalent momentum transfer problem, a fluid in a cylindrical container is rotating with constant velocity $u_\theta = r\Omega$. At $t = 0$, the wall of the container at $r = R$ is instanteneously brought to rest, $u_\theta = 0$. We would like to find out the velocity profile within the cylinder. The scaled velocity field is defined as $u_\theta^* = (u_\theta / r\Omega)$.

3.5 Effect of pressure on momentum transport

The momentum balance condition states that the rate of change of momentum is equal to the applied force.

$$\left[\begin{array}{c}\text{Rate of}\\ \text{momentum in}\end{array}\right]-\left[\begin{array}{c}\text{Rate of}\\ \text{momentum out}\end{array}\right]+\left[\begin{array}{c}\text{Sum of forces}\\ \text{acting on the system}\end{array}\right]=0 \tag{3.226}$$

This balance equation is written for a control volume of fluid which is in the form of a thin shell. Momentum enters or leaves the control volume due to fluid flow into or out of the control volume, or due to the stresses acting on the surface due to viscosity. The forces acting on the system are usually the gravitational force or the centrifugal force. Momentum balances are usually easy to apply only if the streamlines are straight. Applying momentum balances to systems with curved streamlines is more difficult, as we shall see in the last example of this section.

The procedure for solving problems with momentum balance is to write the momentum balance equation for a shell of finite thickness, and then let the thickness go to zero. In this limit, the difference equations for the velocity field across a finite shell reduces to differential equations for the velocity fields. These can then be solved, subject to appropriate boundary conditions, in order to determine the velocity fields.

The boundary conditions generally involve specifying the velocity or stress fields at the boundaries of the flow. These conditions depend on the surface adjoining the liquid at its boundaries.

1. If there is a solid surface adjacent to the fluid, the appropriate boundary condition is the 'no slip' condition which states that the velocity of the fluid at the surface is equal to the velocity of the surface itself.

2. At a liquid - gas interface, the momentum flux, and consequently the velocity gradient, in the liquid side is assumed to be zero, because the viscosity of the gas is small compared to that of the liquid.

3. At a liquid - liquid interface, the momentum flux and the velocity are continuous across the interface.

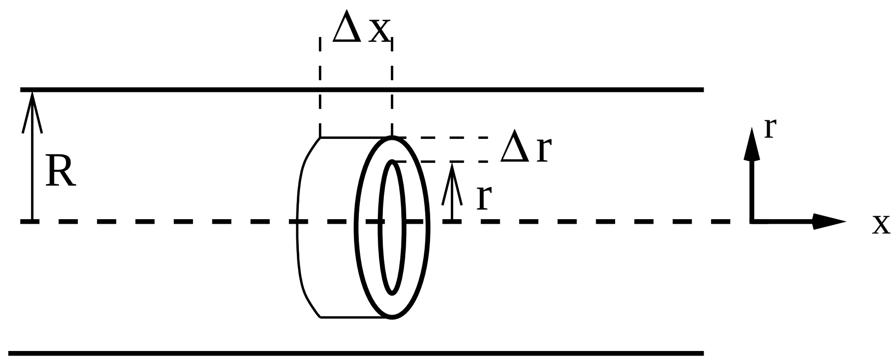


Figure 3.14: Flow in a circular tube.

3.5.1 Flow through a pipe:

The flow through a circular tube is often encountered in engineering applications, and the laminar flow can be analysed using shell momentum balances. The cylindrical coordinate system that is used for the analysis is shown in figure 3.14.

Consider a cylindrical shell of thickness Δr and length Δx. The balance equation for the x component of the momentum can be written as,

$$\begin{array}{c}\text{Rate of change}\\ \text{of } x \text{ momentum}\end{array} = \text{Sum of forces } l \tag{3.227}$$

The rate of change of momentum in a time interval Δt within the differential volume under consideration can be written as,

$$\begin{array}{c}\text{Rate of change}\\ \text{of } x \text{ momentum}\end{array} = \frac{\Delta(\rho u_x)}{\Delta t}(2\pi r \Delta r \Delta z \tag{3.228}$$

where $\Delta(\rho u_x)$ is the change in the momentum per unit volume in the time interval Δt.

There are four bounding surfaces for the differential volume under consideration, two or which are at perpendicular to the x axis and located at x and $x + \Delta x$, and two of which are perpendicular to the radial co-ordinate and are located at r and $r + \Delta r$. The forces acting on the surfaces at x and $x + \Delta x$ can be separated into two parts, the first due to the pressure acting on the surfaces, and the second due to the flux of momentum due to fluid motion. The forces due to fluid pressure can be written as,

$$\begin{aligned}\text{Force due to pressure}\atop\text{on surface at } x &= p(r,\theta,x)(2\pi r\Delta r)\\ \text{Force due to pressure}\atop\text{on surface at } x+\Delta x &= -p(r,\theta,x+\Delta x)(2\pi r\Delta r)\end{aligned} \tag{3.229}$$

Note that there is a negative sign for the force at $x+\Delta x$, because the pressure always acts along the inward unit normal at the surface, and the inward unit normal at $x+\Delta x$ is in the negative x direction. There is an additional force due to the flow of momentum into the differential volume through the surface at x and the flow of momentum out of the differential volume through the surface at $x+\Delta x$. This force is given by the product of the momentum flux (momentum transported per unit area per unit time) and the surface area. The momentum flux is the product of the momentum density (ρu_x) (per unit volume) and the normal velocity to the surface u_x. This is analogous to the mass flux, which is the product of the concentration (mass density) c and the normal velocity. Therefore, the force due to the flow of momentum is,

$$\begin{aligned}\text{Force due to momentum flow}\atop\text{on surface at } x &= ((\rho u_x)u_x(2\pi r\Delta r)\\ \text{Force due to momentum flow}\atop\text{on surface at } x+\Delta x &= -p(r,\theta,x+\Delta x)(2\pi r\Delta r)\end{aligned} \tag{3.230}$$

Note that the force at $(x+\Delta x)$ is negative, since momentum leaves the differential volume at this surface. We can now add up all the contributions to get

$$-2\pi rL\tau_{rz}|_r+2\pi rL\tau_{rz}|_{r+\Delta r}+2\pi r\Delta r\rho v_z^2\Big|_{z=0}-2\pi r\Delta r\rho v_z^2\Big|_{z=L}+2\pi r\Delta rL\rho g+2\pi r\Delta r(p(r,x,t)-p(r,x+\Delta x,t) \tag{3.231}$$

Since the fluid is considered to be incompressible, v_z is equal at $z=0$ and $z=L$. Therefore, the momentum flux due to fluid motion cancels out, and we can divide by $2\pi L\Delta r$ and take the limit $\Delta r\to 0$ to get

$$\rho\frac{\partial u_x}{\partial t}=\frac{1}{r}\frac{\partial(r\tau_{rz})}{\partial r}-\frac{\partial p}{\partial x} \tag{3.232}$$

Using Newton's law of viscosity, $\tau_{zr}=\mu(\partial u_z/\partial r)$, we obtain the momentum conservation equation,

$$\rho\frac{\partial u_z}{\partial t}=-\frac{\partial p}{\partial z}+\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) \tag{3.233}$$

Steady flow:

At steady state, the momentum conservation equation in the pipe reduces to,

$$-\frac{\partial p}{\partial z} + \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) \tag{3.234}$$

The boundary conditions are the no-slip condition at the wall of the pipe ($r = R$),

$$u_z = 0 \quad \text{at} \quad r = R \tag{3.235}$$

and the symmetry condition at the axis of the tube,

$$\frac{\partial u_z}{\partial r} = 0 \quad \text{at} \quad r = 0 \tag{3.236}$$

The above equation can be integrated quite easily of the pressure is independent of r (no pressure variation in the radial direction). We will see later, when we derive the momentum balance equation in the r direction, that the pressure variation in the r diretion is, indeed, zero, when there is no radial velocity u_r. Physically, the reasoning is as follows. Just as the axial momentum balance equation 3.233 balances the shear stresses (due to the variation of u_z in the radial direction), the acceleration term on the left (due to the time derivative of u_z on the left) and the axial pressure gradient, the radial momentum balance equation will balance the stresses in the radial direction (due to the variation of u_r in the radial and axial directions), the acceleration term (due to the time derivative of u_r) and the radial pressure gradient. However, u_r is identically zero throughout the pipe, because flow is only in the axial (z) direction. Therefore, the pressure is invariant in the radial direction, and $(\partial p/\partial r) = 0$.

Since the pressure is only a function of z, equation 3.234 can be integrated to obtain,

$$u_z = \frac{r^2}{4\mu}\frac{\partial p}{\partial z} + C_1 \log{(r)} + C_2 \tag{3.237}$$

The boundary condition 3.236 is satisfied only if $C_1 = 0$, and $C_2 = -(R^2/4\mu)\frac{\partial p}{\partial z}$ from boundary condition 3.235. Therefore, the final expression for the velocity field is,

$$u_z = -\frac{1}{4\mu}\frac{\partial p}{\partial z}(R^2 - r^2) \tag{3.238}$$

This is the familiar parabolic velocity profile for the flow in a tube, called the Hagen-Poiseuille flow. Note that the velocity is positive when the pressure gradient is negative, that is, when the pressure decreases along the z direction. The maximum value of the velocity at the center of the tube is,

$$u_{max} = -\frac{R^2}{4\mu}\frac{\partial p}{\partial z} \tag{3.239}$$

The velocity profile 3.238 can also be expressed in terms of the maximum velocity as,

$$u_z = u_{max}\left(1 - \left(\frac{r}{R}\right)^2\right) \tag{3.240}$$

The shear stress is,

$$\begin{aligned} \tau_{zr} &= \mu\frac{\partial u_z}{\partial r} \\ &= -\frac{2\mu u_{max} r}{R^2} \end{aligned} \tag{3.241}$$

The shear stress is zero at the center of the tube, and it increases linearly with radius. At the wall, the wall shear stress is given by,

$$\tau_{zr}|_{r=R} = -\frac{2\mu u_{max}}{R} \tag{3.242}$$

Note that the above shear stress is negative because it is the force per unit area at a surface whose outward unit normal is in the radial direction. At the wall, this represents the force per unit area exerted on the fluid by the wall, which is in the $-z$ direction. There is a force of equal magnitude exerted by the fluid on the wall, which is in the $+z$ direction in accordance with Newton's third law.

The volumetric flow rate is the product of the flow velocit and the cross-sectional area of the tube. In this case, the velocity is varying in the r direction, and so it is necessary to integrate the velocity $v_z(r)$ times the differential area of a circular strip, $(2\pi r dr)$ (figure **??**) from $r = 0$ to $r = R$.

$$\begin{aligned} Q &= \int_0^R u_z 2\pi r\, dr \\ -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial z} & \end{aligned} \tag{3.243}$$

The average velocity $\bar{u}$ is defined as the ratio of the flow rate Q and the cross-sectional area (πR^2),

$$\bar{u} = \frac{Q}{\pi R^2} = -\frac{R^2}{8\mu}\frac{\partial p}{\partial z} \tag{3.244}$$

The maximum velocity equation 3.240 is two times the average velocity 3.244,

$$u_{max} = 2\bar{u} \tag{3.245}$$

The friction factor for the flow through a pipe is defined as the ratio of the wall shear stress and $(\rho\bar{u}^2/2)$,

$$f = \frac{\tau_{rz}\big|_{r=R}}{(\rho\bar{u}^2/2)} \tag{3.246}$$

Using equation 3.242 for the wall shear stress, we find that the friction factor is,

$$f = \frac{2\mu u_{max}}{R(\rho\bar{u}^2/2)} \tag{3.247}$$

Using equation 3.245 for the relation between u_{max} and $\bar{u}$, we find that the friction factor is,

$$f = \frac{16\mu}{\rho u_{max} R} = \frac{16}{Re} \tag{3.248}$$

where $Re = (\rho u_{max} R/\mu)$, the Reynolds number based on the maximum fluid velocity and the radius of the tube. The Reynolds number can also be expressed as $Re = (\rho\bar{u}D/\mu)$, which is the Reynolds number based on the average velocity and the tube diameter, because $\bar{u} = (u_{max}/2)$ and $D = 2R$.

The friction factor correlation 3.248 is valid only when the flow in the tube is in the laminar regime, that is, when the Reynolds number is less than 2100. When the Reynolds number increases beyond 2100, there is a spontaneous transition to a turbulent flow, in which the friction factor is significantly higher than that in a laminar flow. The solution 3.248 for a laminar flow is a valid solution for the momentum balance equation even when the Reynolds number is higher than 2100. However, this solution becomes *unstable* when the Reynolds number is higher than 2100, and naturally occuring disturbances in the flow cause a spontaneous transition to a more complicated 'turbulent' velocity profile. The distinction between the two is as follows. In the laminar flow, the streamlines within the flow are straight and parallel to each other, and there is no motion in the cross-stream directon. The transport of momentum in the cross-stream (radial) direction is by momentum diffusion, caused by the fluctuating velocity of the molecules. We had discussed this mechanism in detail in chapter 2. In contrast, in a turbulent flow, there are significant velocity fluctuations in both the stream-wise and the cross-stream directions, though the cross-stream velocity is zero on average. The momentum transport is caused by the motion of correlated regions within the fluid called 'eddies'. This mechanism of momentum transport is more efficient than the molecular mechanism, and so this results in a higher friction factor than that in a laminar flow. Due to this efficient cross-stream momentum transport, the velocity profile in a turbulent flow is also much flatter and plug-like than the parabolic velocity profile in a laminar flow.

Oscillatory flow in a pipe

Oscillatory flows in tubes are of practical importance in physiological fluid dynamics, where the pumping of the heart causes a periodic variation in the flow velocity along the arteries. Here, we study the simple example of the velocity profile due to an oscillatory pressure gradient, $(\delta p/\delta z) = \tilde{p}\cos(\omega t)$. However, the solution procedure can be used for a more general time-periodic variation with frequency ω, because any periodic modulation can be expressed as a sum of sine waves with frequencies that are integer multiples of ω.

We consider an unsteady (time-periodic) but fully developed flow along the tube, where the velocity is dependent on time but is independent of the z co-ordinate. The momentum conservation equation 3.233 for the present case is,

$$\rho\frac{\partial u_z}{\partial t} = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) - K\cos(\omega t) \tag{3.249}$$

The boundary conditions at the center and the wall of the tube are the same as those for a steady flow, 3.235 and 3.236. It is natural to define a scaled radial co-ordinate as $r^* = (r/R)$. The scaled time can be defined as $t^* = (\omega t)$, since ω is the frequency of the pressure wave. How do we scale the velocity u_z? It is clear that the magnitude of the velocity is related to K, which is the amplitude of the pressure fluctuations. Therefore, we express equation

3.249 in terms of r^* and t^*, and divide throughout by K, to obtain,

$$\frac{\rho\omega}{K}\frac{\partial u_z}{\partial t^*} = \frac{\mu}{KR^2}\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u_z}{\partial r^*}\right) - \cos(t^*) \tag{3.250}$$

It is clear that all terms in the above equation are dimensionless, and therefore, we can define a scaled velocity either as $u_z^* = (\mu u_z/KR^2)$, or as $u_z^* = (\rho\omega u_z/K)$. The former is obtained by balancing the viscous and pressure forces, and is appropriate for low Reynolds numbers where inertial forces are negligible. The latter is obtained by balancing inertial and pressure forces, and is appropriate when the Reynolds number is large so that viscous forces are negligible. We will proceed using the scaling obtained by balancing the viscous and pressure forces, and return later to consider the situation when the Reynolds number is large. Using a scaled velocity $u_z^* = (\mu u_z/KR^2)$, equation 3.250 becomes,

$$\mathrm{Re}_\omega\frac{\partial u_z^*}{\partial t^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u_z^*}{\partial r^*}\right) - \cos(t^*) \tag{3.251}$$

where $\mathrm{Re}_\omega = (\rho\omega R^2/\mu)$ is the Reynolds number based on the frequency of oscillations and the tube radius.

As in the case of an oscillatory flow past a flat plate in section **??**, it is more convenient to solve the equation 3.251 with an inhomogenous term of the form $\exp(\imath t)$, and then take the real part of this to obtain the solution for u_z^*. Therefore, we define a complex velocity field $u_z^\dagger$ as the solution of the equation,

$$\mathrm{Re}_\omega\frac{\partial u_z^\dagger}{\partial t^*} = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u_z^\dagger}{\partial r^*}\right) - \exp(\imath t^*) \tag{3.252}$$

with boundary conditions

$$u_z^\dagger = 0 \quad \text{at} \quad r^* = 1 \tag{3.253}$$

$$\frac{\partial u_z^\dagger}{\partial r^*} = 0 \quad \text{at} \quad r^* = 0 \tag{3.254}$$

Clearly, equation 3.251 is the real part of equation 3.252, and the boundary conditions 3.235 and 3.236 are the real parts of the boundary conditions 3.253 and 3.254. Therefore, the solution u_z^* is the real part of the complex velocity $u_z^\dagger$.

Since the equation 3.252 is a linear inhomogeneous equation for $u_z^\dagger$ with a time-periodic forcing proportional to $\exp(\imath t^*)$, the solution $u_z^\dagger$ is also periodic in time with the sale frequency,

$$u_z^\dagger = \tilde{u}_z(r)\exp(\imath t) \tag{3.255}$$

where $\tilde{u}_z(r)$ is only a function of the radial co-ordinate. Substituting the solution 3.255 into the equation 3.252, and dividing the resulting equation by the common factor $\exp(\imath t^*)$, the equation for $\tilde{u}_z$ becomes

$$\left(\frac{1}{r^*}\frac{d}{dr^*}r^*\frac{d\tilde{u}_x}{dr^*}\right) - \imath \mathrm{Re}_\omega \tilde{u}_z = 1 \tag{3.256}$$

The solution for $\tilde{u}_x$ for the inhomogeneous equation 3.256 can be divided into a general and a particular solution, $\tilde{u}_z = \tilde{u}_z^g + \tilde{u}_z^p$. The homogeneous solution is the solution of the equation,

$$\left(\frac{1}{r^*}\frac{d}{dr^*}r^*\frac{d\tilde{u}_z^g}{dr^*}\right) - \imath \mathrm{Re}_\omega \tilde{u}_z^g = 0 \tag{3.257}$$

while the particular solution is any one solution of the inhomogeneous equation 3.256.

The simplest particular solution, which satisfies the equation 3.256, is just a constant, $\tilde{u}_z^p = -\imath \mathrm{Re}_\omega^{-1}$. Equation 3.257 for the general solution can be solved using the substitution $r^\dagger = \sqrt{-\imath \mathrm{Re}_\omega} r^*$, to obtain,

$$r^{\dagger 2}\frac{\partial^2 \tilde{u}_z^g}{\partial r^{\dagger 2}} + r^\dagger \frac{\partial \tilde{u}_z^g}{\partial r^\dagger} + r^{\dagger 2}\tilde{u}_z^g = 0 \tag{3.258}$$

The above equation is the Bessel equation of zeroeth order discussed earlier, and the solution of this equation is,

$$\tilde{u}_z^g = C_1 J_0(\sqrt{-\imath \mathrm{Re}_\omega} r^*) + C_2 Y_0(\sqrt{-\imath \mathrm{Re}_\omega} r^*) \tag{3.259}$$

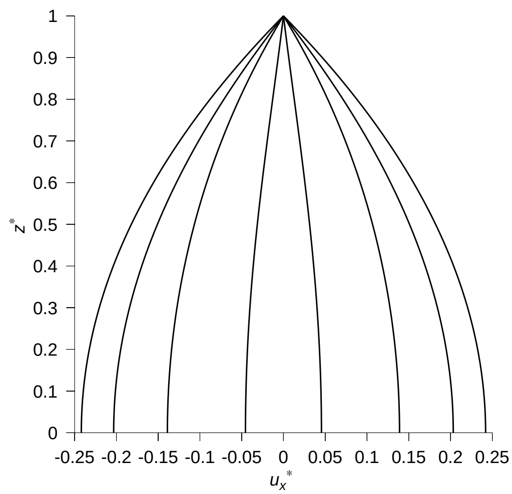
where J_0 and Y_0 are the Bessel functions of zeroeth order, and C_1 and C_2 are constants. Therefore, the final solution of equation 3.256 is,

$$\tilde{u}_z = C_1 J_0(\sqrt{-\imath \mathrm{Re}_\omega} r^*) + C_2 Y_0(\sqrt{-\imath \mathrm{Re}_\omega} r^*) - \imath \mathrm{Re}_\omega^{-1} \tag{3.260}$$

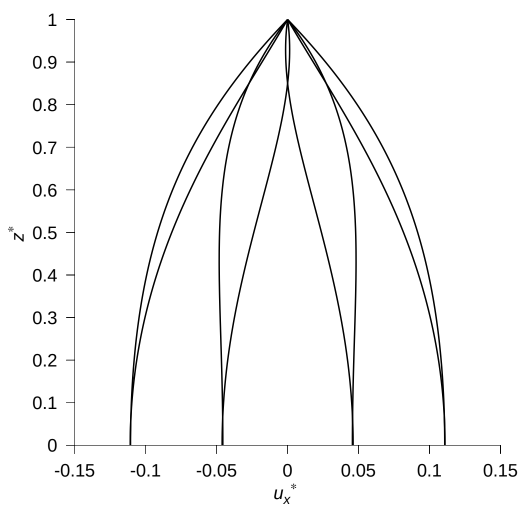
The constants C_1 and C_2 in equation 3.256 are evaluated from the boundary conditions 3.253 and 3.254. The constant C_2 is zero form the boundary condition 3.253, because Y_0 goes to $-\infty$ in the limit $r^* \to 0$. The constant C_1, evaluated from the boundary condition 3.254, is $C_1 = \imath(\mathrm{Re}_\omega J_0(\sqrt{-\imath\mathrm{Re}_\omega}))^{-1}$. Therefore, the final solution for $\tilde{u}_z$ is of the form,

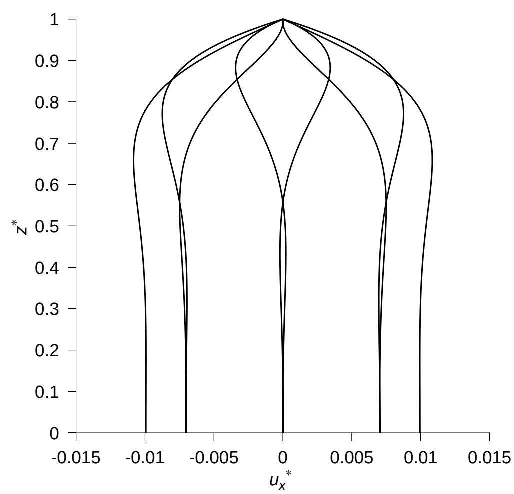
$$\tilde{u}_z = \imath\mathrm{Re}_\omega^{-1}\,(1-$$

(a)



(b)





(c) (d)

The velocity u_z^*, equation 3.121, as a function of r^*, for $\mathrm{Re}_\omega = 0.1$ (a), $\mathrm{Re}_\omega = 1.0$ (b), $\mathrm{Re}_\omega = 10.0$ (c) and $\mathrm{Re}_\omega = 100.0$ (d). The profiles, form right to left, are at $t^* = 0$, $t^* = (\pi/4)$, $t^* = (\pi/2)$, $t^* = (3\pi/4)$ and $t^* = \pi$.

One way to obtain the solution in the limit of low Reynolds number, we can use a Taylor series expansion of the solution 3.262 in the small parameter Re_ω. Another method is to use an expansion of the equation 3.256 itself. The latter is useful even in more complicated situations where we are not able to obtain an analytical solution for the velocity field, and it provides our first encounter with the techniques of 'regular' and 'singular' perturbation expansions. Therefore, we go through the analysis of the equation 3.256 in the limits $\mathrm{Re}_\omega \ll 1$ and $\mathrm{Re}_\omega \gg 1$ in some detail.

In the limit $\mathrm{Re}_\omega \ll 1$, we can obtain an approximate solution by considering the limit $\mathrm{Re}_\omega \to 0$ in equation 3.256. The solution $\tilde{u}_z$ in a series in Re_ω,

$$\tilde{u}_z = \tilde{u}_z^{(0)} + \mathrm{Re}_\omega \tilde{u}_z^{(1)} + \mathrm{Re}_\omega^2 \tilde{u}_z^{(2)} + \ldots \tag{3.262}$$

This expansion is inserted into equation 3.256, to obtain,

$$\imath \mathrm{Re}_\omega (\tilde{u}_z^{(0)} + \mathrm{Re}_\omega \tilde{u}_z^{(1)} + \mathrm{Re}_\omega^{(2)}) = \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial}{\partial r^*}(\tilde{u}_z^{(0)} + \mathrm{Re}_\omega \tilde{u}_z^{(1)} + \mathrm{Re}_\omega^2 \tilde{u}_z^{(2)})\right) - 1 \tag{3.263}$$

We collect the coefficients of Re_ω^0, Re_ω^1, Re_ω^2, $\ldots$ in equation 3.263 to obtain,

$$\begin{aligned} & \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial \tilde{u}_z^{(0)}}{\partial r^*}\right) - 1 + \\ \mathrm{Re}_\omega & \left(\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial \tilde{u}_z^{(1)}}{\partial r^*}\right) - \imath \tilde{u}_z^{(0)}\right) + \\ \mathrm{Re}_\omega^2 & \left(\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial \tilde{u}_z^{(2)}}{\partial r^*}\right) - \imath \tilde{u}_z^{(1)}\right) + \\ & \ldots = 0 \end{aligned} \tag{3.264}$$

In the limit $\mathrm{Re}_\omega \to 0$, the first term on the left side of equation 3.264 is $O(1)$ (independent of Re_ω), the second is $O(\mathrm{Re}_\omega)$ (proportional to Re_ω), the third term on the left is $O(\mathrm{Re}_\omega^2)$ (proportional to Re_ω^2), in the limit $\mathrm{Re}_\omega \to 0$. Here the symbol $O(\mathrm{Re}_\omega^i)$ (called 'order of Re_ω power i') indicates that that term goes to zero as the i^{th} power of the small parameter Re_ω in the limit $\mathrm{Re}_\omega \to 0$. If equation 3.264 is to be satisfied in the limit $\mathrm{Re}_\omega \to 0$, then the individual coefficients of of $\mathrm{Re}_\omega^0, \mathrm{Re}_\omega^1, \mathrm{Re}_\omega^2, \ldots$ on the left side of equation

3.264 should all be equal to zero. Therefore, we obtain a series of equations,

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial \tilde{u}_z^{(0)}}{\partial r^*}\right) - 1 = 0 \tag{3.265}$$

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial \tilde{u}_z^{(1)}}{\partial r^*}\right) - \imath\tilde{u}_z^{(0)} = 0 \tag{3.266}$$

$$\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial \tilde{u}_z^{(2)}}{\partial r^*}\right) - \imath\tilde{u}_z^{(1)} = 0 \tag{3.267}$$

It is necessary to expand the boundary conditions also in a series in Re_ω. In the present example, this expansion is quite easy, because the boundary conditions 3.253 and 3.254 are homogeneous. Therefore, the boundary conditions are of the form,

$$\tilde{u}_z^{(i)} = 0 \quad \text{at} \quad r^* = 1 \tag{3.268}$$

and

$$\frac{\partial \tilde{u}_z^{(i)}}{\partial r^*} = 0 \quad \text{at} \quad r^* = 0 \tag{3.269}$$

for $i = 0, 1, 2, \ldots$.

Equations 3.265, 3.266, and 3.267 can be solved sequentially to obtain solutions for $\tilde{u}_z^{(0)}$, $\tilde{u}_z^{(1)}$, $\tilde{u}_z^{(2)}$. The solutions are,

$$\tilde{u}_z^{(0)} = -\frac{(1 - r^{*2})}{4} \tag{3.270}$$

$$\tilde{u}_z^{(1)} = \frac{\imath(3 - 4r^{*2} + r^{*4})}{64} \tag{3.271}$$

$$\tilde{u}_z^{(2)} = \frac{(19 - 27r^{*2} + 9r^{*4} + r^{*6}_{\,)}}{2304} \tag{3.272}$$

Therefore, the final solution for u_z^* in the limit $\mathrm{Re}_\omega \to 0$, obtained by multiplying $\tilde{u}_z$ (equations 3.270, 3.271 and 3.272) by $\exp(\imath t^*)$ and taking the real part, is,

$$\begin{aligned} u_z^* \;=\; & -\frac{(1 - r^{*2})\cos(t^*)}{4} - \frac{\mathrm{Re}_\omega(3 - 4r^{*2} + r^{*4})\sin(t^*)}{64} \\ & + \frac{\mathrm{Re}_\omega^2(19 - 27r^{*2} + 9r^{*4} + r^{*6}_{\,)}\cos(\theta)}{2304} + \ldots \end{aligned} \tag{3.273}$$

Thus, this procedure, referred to as a 'regular perturbation' expansion, provides a solution for the velocity field as a series in the small parameter Re_ω.

For zero Reynolds number, the velocity field is,

$$u_z^* = -\frac{(1 - r^{*2})}{4} \cos(t^*) \tag{3.274}$$

or the dimensional velocity field u_z is,

$$u_z = -\frac{K \cos(\omega t) R^2}{4\mu} \left(1 - \frac{r^2}{R^2}\right) \tag{3.275}$$

This solution is identical to the steady solution 3.238, with the pressure gradient given by $(K \cos(\omega t))$, which is the instantaneous value of the pressure gradient at time t. The steady solution is recovered because the inertial term on the left side of equation 3.233 has been neglected in the limit of small Reynolds number. Physically, the Reynolds number $\mathrm{Re}_\omega = (\omega R^2/\nu)$ can be considered as the ratio of two time scales, the time period of oscillation of the pressure field ω^{-1} and the time scale for momentum diffusion over a distance R, which is (R^2/ν). When the Reynolds number is small, the time scale for momentum diffusion is small compared to the period of oscillation of the pressure field. Therefore, the velocity field responds instanteneously to the change in the pressure, and we obtain a solution that is identical to the steady solution for the instanteneous value of the pressure gradient.

Next, we consider the limit $\mathrm{Re}_\omega \gg 1$. In the high Reynolds number limit, it is necessary to scale the velocity by the inertial scale $(\rho\omega/K)$ in equation 3.250, since we would expect a balance between the inertial forces and the pressure gradient when the viscous forces are small compared to the inertial forces. The non-dimensional velocity is defined as $u_z^{**} = (\rho u_z \omega/K)$, and the equation 3.250 becomes,

$$\frac{\partial u_z^{**}}{\partial t^*} = \frac{1}{\mathrm{Re}_\omega} \frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial u_z^{**}}{\partial r^*}\right) - \cos(t^*) \tag{3.276}$$

The substitution

$$u_z^{**} = \mathrm{Real} \tilde{u}_z^{**} \exp(\imath t^*) (3.277)$$

is used to obtain an ordinary differential equation for $\tilde{u}^{**}$,

$$\imath \tilde{u}_z^{**} = \frac{1}{\mathrm{Re}_\omega} \frac{1}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial \tilde{u}_z^{**}}{\partial r^*}\right) - 1 \tag{3.278}$$

A naive approximation in this limit would be to just neglect the viscous term on the right in comparison to the inertial term on the left in equation 3.278, to obtain,

$$\imath \tilde{u}_z^{**} = -1 \tag{3.279}$$

This provides the solution for the velocity field as,

$$\tilde{u}_z^{**} = \imath \tag{3.280}$$

This solution has to satisfy the boundary conditions 3.253 and 3.254. Boundary condition 3.254 is satisfied, but it is clear that we cannot satisfy boundary condition 3.253, because we do not have any constants in the solution 3.280! Why?

The mathematical reason can be explained by examining the approximation made in going from equation 3.278 to 3.279. Equation 3.278 is a second order differential equation in r^*, and so the solution of this equation will contain two constants which are fixed by the boundary conditions. However, in going from equation 3.278 to 3.279, we neglected the highest derivative, because of the small factor Re_ω^{-1}. In doing so, we have converted a differential equation into an algebraic equation, and there are no constants in the solution.

The physical reason for the inability to satisfy the no-slip condition at the pipe wall is as follows. The velocity at the wall will decrease to zero only if the wall exerts a stress on the fliud, that is, if there is momentum diffusion from the wall to the fluid. In going from equation 3.278 to 3.279, we have neglected the momentum diffusion, and so there is no shear stress exerted by the pipe wall on the fluid. Due to this, we are not able to satisfy the no-slip boundary condition. However, in the real pipe flow, there is a no-slip condition at the surface which is satisfied by the actual flow. How can this paradox be resolved?

We had neglected the momentum diffusion term in equation 3.279 because the Reynolds number Re_ω is small. The Reynolds number, $\mathrm{Re}_\omega = (\omega R^2/\nu)$, is the ratio of the time required for diffusion over a lenght scale R, $\sim (\nu/R^2)$, and the period of oscillation, $\sim \omega^{-1}$. Even when this number is large, diffusion is still taking place, but the length scale to which momentum diffuses is smaller than R. Therefore, there is an 'inner length scale' at the wall of the pipe, which is the length scale over which diffusion takes place. Over a time period $\sim \omega^{-1}$, this length scale is obviously $(\nu/\omega)^{1/2} \sim (R\mathrm{Re}_\omega^{-1/2})$. Therefore, if we scale the distance from the wall by this inner length scale, there will be a balance between the inertial and the viscous terms in the momentum balance equation 3.278.

Mathematically, this is accomplished by postulating a 'boundary layer' at the wall with thickness δR, where $\delta \ll 1$. The condition for finding the value of δ is that in the limit $\mathrm{Re}_\omega \to \infty$ (as we keep increasing Re_ω), there continues to be a balance between the inertial and viscous terms in the conservation equation 3.276, within the region of thickness δR. In order to apply this condition, we focus on a thin layer near the wall, and define the distance from the wall, $(R - r) = \delta R y$, where y is the inner co-ordinate. Therefore, the inner co-ordinate is defined as,

$$y = \delta^{-1}(1 - r^*) \tag{3.281}$$

The inner co-ordinate is defined in such a manner that y is $O(1)$ within the region where there is a balance between inertial and viscous forces in the limit $\mathrm{Re}_\omega \to \infty$. We return to the dimensional equation 3.276, and express r^* in terms of the scaled co-ordinate y,

$$\frac{\partial u^{**}}{\partial t^*} = \frac{1}{\mathrm{Re}_\omega} \frac{1}{(1 + \delta y)} \frac{1}{\delta} \frac{\partial}{\partial y} \left((1 + \delta y) \frac{1}{\delta} \frac{\partial u^{**}}{\partial y} \right) - \cos(t^*) \tag{3.282}$$

In the limit $\delta \ll 1$, we can neglect terms proportional to δ to obtain,

$$\frac{\partial u^{**}}{\partial t^*} = \frac{1}{\mathrm{Re}_\omega \delta^2} \frac{\partial}{\partial y} \left(\frac{\partial u^{**}}{\partial y} \right) - \cos(t^*) \tag{3.283}$$

It is clear that, for the inertial and viscous terms to be of the same magnitude in the limit $\mathrm{Re}_\omega \to \infty$, we require that $\delta \sim \mathrm{Re}_\omega^{-1/2}$. The thickness of the 'boundary layer' near the wall, where viscous and inertial effects are of the same magnitude, is $(\delta R) = (\omega/\nu)^{1/2}$, as anticipated above on the basis of physical arguments.

One could set δ equal to some constant multiplied by $\mathrm{Re}_\omega^{-1/2}$, and proceed to solve the problem; with this choice, the inertial and viscous terms are comparable so long as the constant is $O(1)$. The question arises, what should the value of the constant be? The answer is that the constant could be any value; while the solution of equation 3.283 in terms of the scaled co-ordinate y will depend on C, the solution in terms of the original co-ordinate r^* will be independent of this constant. In order to illustrate this, we set will use $\delta = C\mathrm{Re}_\omega^{-1/2}$ without specifying C in the present problem, and then show that the final solution does not depend on C. When we use this procedure in our later analysis of boundary layer theory, we will just use the simplest choice $C = 1$.

Substituting $\delta = C\mathrm{Re}_\omega^{-1/2}$ in equation 3.283, we obtain,

$$\frac{\partial u^{**}}{\partial t^*} = \frac{1}{C}\frac{\partial}{\partial y}\left(\frac{\partial u^{**}}{\partial y}\right) - \cos(t^*) \tag{3.284}$$

As usual, we use the substitution $u^{**} = \mathrm{Real}(\tilde{u}^{**}(y)\exp(\imath t^*))$, to obtain an equation for $\tilde{u}^{**}(y)$,

$$\imath(y) = \frac{1}{C}\frac{\partial^2}{\partial y^2} - 1 \tag{3.285}$$

This is easily solved to obtain,

$$\tilde{u}^{**} = \imath + C_1 \exp(\sqrt{\imath C} y) + C_2 \exp -(\sqrt{\imath C} y) \tag{3.286}$$

The constants in equation 3.286 are evaluated from the boundary conditions, which are to be re-expressed in terms of the scaled co-ordinate y. The boundary condition 3.253, expressed in terms of y, is

$$\tilde{u}^{**} = 0 \quad \text{at} \quad y = 0 \tag{3.287}$$

The second boundary condition 3.254 is applied at $r^* = 0$ which is equivalent to $y = \delta^{-1}$. In the limit $\delta \to 0(\mathrm{Re}_\omega \to \infty)$, this is equivalent to,

$$\frac{\partial \tilde{u}^{**}}{\partial y} = 0 \quad \text{for} \quad y \to \infty \tag{3.288}$$

Using these two conditions, the constant $C_1 = 0$ and $C_2 = -\imath$ in equation 3.286, and therefore, the final solution for $\tilde{u}^{**}$ is,

$$\tilde{u}^{**} = \imath(1 - \exp(-\sqrt{\imath C} y)) \tag{3.289}$$

Substituting $y = (1 - r^*)/\delta$ and $\delta = \sqrt{C\mathrm{Re}_\omega}$, the solution for $\tilde{u}^{**}$ in terms of r^* is,

$$\tilde{u}^{**} = \imath(1 - \exp(-\sqrt{\imath \mathrm{Re}_\omega^{-1/2}}(1 - r^*)) \tag{3.290}$$

As we had anticipated earlier, the solution for $\tilde{u}^{**}$ in terms of r^* is independent of the constant C used in the definition of δ. Therefore, without loss of generality, we can set this constant to any value. In practice, it is most convenient to set this equal to 1.

The solution for u^{**} is,

$$\begin{aligned} u^{**} &= \mathrm{Real}(\tilde{u}^{**}) \\ &= -\sin(t^*)(1 - \exp(-(1 - r^*)/\sqrt{2\mathrm{Re}_\omega})\cos((1 - r^*)/\sqrt{2\mathrm{Re}_\omega})) + \cos(t^*)\sin((1 - r^*)/\sqrt{} \end{aligned}$$

3.6 Spherical co-ordinates:

The spherical co-ordinate system is used to analyse transport in or around objects with spherical symmetry, such as spherical particles, bubbles or drops immersed in a fluid. If we were analysing the mass or energy transport from a spherical particle immersed in a fluid, it is preferable to have a co-ordinate system in which the surface of the particle is a surface on which one of the co-ordinates is a constant, so that boundary conditions can be easily applied there. The surface of a sphere in a Cartesian co-ordinate system is described by the equation $x^2+y^2+z^2 = R^2$; clearly, this is not a convenient description for applying boundary conditions. In the spherical co-ordinate system, shown in figure **??**, the three co-ordinates are the radius r, the distance from the origin, the azimuthal angle θ, which is the angle made by the position vector with the z axis, and meridional angle ϕ, the angle made by the (projection of the position vector on the $x-y$ plane) with the x axis. Surfaces of constant r are now spherical surfaces with radius r.

3.6.1 Balance equation:

Here, we consider one-dimensional transport, in which there is a variation of the concentration or temperature fields only in the radial direction. In order to obtain a balance equation, we consider a spherical shell bounded by two surfaces at r and $r+\Delta r$. The volume of the shell is $4\pi r^2 \Delta r$, while the surface area is $4\pi r^2$. The terms in the mass balance equation, 3.1, for this shell, are as follows. The accumulation of mass in the shell in a time Δt is,

$$\begin{pmatrix} \text{Accumulation of mass} \\ \text{in the shell} \end{pmatrix} = (c(r,t+\Delta t) - c(r,t)) 4\pi r^2 \Delta r \qquad (3.292)$$

The total mass entering the shell at r is the product of the mass flux, the surface area, and the time interval Δt,

$$\begin{pmatrix} \text{Input of} \\ \text{mass into shell} \end{pmatrix} = -\left(j_r(4\pi r^2 \Delta t)\right)\Big|_r \qquad (3.293)$$

where j_r is the mass flux in the radial direction. Similarly, the total mass leaving the shell at $r+\Delta r$ is

$$\begin{pmatrix} \text{Output of} \\ \text{mass from shell} \end{pmatrix} = \left(j_r(4\pi r^2 \Delta t)\right)\Big|_{r+\Delta r} \qquad (3.294)$$

The source of mass in the differential volume is,

$$\left(\begin{array}{c} \text{Source of} \\ \text{mass in shell} \end{array} \right) = S(4\pi r^2 \Delta r \Delta t) \tag{3.295}$$

where S is the amount of mass generated per unit volume per unit time.

Substituting these into the balance condition **??**, and dividing thoughout by $4\pi r^2 \delta r \delta t$, we obtain,

$$\frac{(c(r,t+\Delta t) - c(r,t))}{\Delta t} = \frac{1}{r^2 \Delta r} \left((r^2 j_r)\Big|_r - (r^2 j_r)\Big|_{r+\Delta r} \right) + S \tag{3.296}$$

Note that, as in the case of the cylindrical co-ordinate system, the surface area $4\pi r^2$ is a function of the radius r, and this surface area is different for the surfaces at r and $r + \Delta r$. Therefore, the factors r^2 in the numerator and denominator of the first two terms on the right side of equation 3.296 cannot be cancelled. Taking the limit $\Delta r \to 0$ and $\Delta t \to 0$, the partial differential equation for the concentration field is

$$\frac{\partial c}{\partial t} = -\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 j_r \right) + S \tag{3.297}$$

The mass flux j_r is related to the concentration gradient in the radial direction by the Fourier's law for heat conduction,

$$j_r = -D \frac{\partial c}{\partial r} \tag{3.298}$$

With this, the mass balance equation becomes,

$$\frac{\partial c}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 D \frac{\partial c}{\partial r} \right) + S \tag{3.299}$$

When the diffusion coefficient is independent of position, the mass balance equation reduces to,

$$\frac{\partial c}{\partial t} = D \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial c}{\partial r} \right) + S \tag{3.300}$$

The energy balance equation for unidirectional transport in a spherical co-ordinate system, derived in a similar manner, is,

$$\frac{\partial T}{\partial t} = \alpha \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + S_e \tag{3.301}$$

where $\alpha = (k/\rho C_p)$ is the thermal diffusivity. The momentum balance equation in a spherical co-ordinate system is more complicated, and so we defer discussion to a later chapter.

3.6.2 Steady diffusion from a sphere:

A spherical particle of radius R, with temperature T_0 at the surface, is immersed in a fluid in which the temperature is T_∞ far from the particle. The system is at steady state, so that the temperature field does not vary with time. We would like to find out the temperature variation in the fluid around the particle, and the total amount of heat emitted by the particle per unit time.

The scaled radius and temperature are defined as, $r^* = (r/R)$ and $T^* = (T - T_\infty)/(T_0 - T_\infty)$. In terms of these scaled co-ordinates, the steady state balance equation is,

$$\frac{1}{r^{*2}} \frac{d}{dr^*} \left(r^{*2} \frac{dT^*}{dr^*} \right) = 0 \tag{3.302}$$

with boundary conditions,

$$T^* = 0 \quad \text{for} \quad r^* \to \infty \tag{3.303}$$

$$T^* = 1 \quad \text{at} \quad r^* = 1 \tag{3.304}$$

Equation 3.302 is easily solved to obtain,

$$T^* = \frac{C_1}{r^*} + C_2 \tag{3.305}$$

where the constants, C_1 and C_2 are determined from the boundary conditions 3.303 and 3.304. The final solution is,

$$T^* = \frac{1}{r^*} \tag{3.306}$$

The dimensional temperature T is given by,

$$T = T_\infty + \frac{(T_0 - T_\infty)R}{r} \tag{3.307}$$

The heat flux from the surface of the sphere is,

$$\begin{aligned} q_r &= -k \frac{\partial T}{\partial r} \\ &= -\frac{k(T_0 - T_\infty)}{R} \frac{\partial T^*}{\partial r^*} \\ &= \frac{k(T_0 - T_\infty)}{R r^{*2}} \end{aligned} \tag{3.308}$$

An important result is obtained if the temperature field is expressed in terms of the total heat generated by the spherical surface, Q, instead of the temperature at the surface. The total heat passing through any spherical shell of radius r is the product of the heat flux and the radius of the shell,

$$\begin{aligned} Q &= 4\pi r^2 q_r \\ &= 4\pi R k (T_0 - T_\infty) \end{aligned} \tag{3.309}$$

From the energy balance condition, this is independent of the radius of the shell, since there is no generation or absorption of energy within the fluid. Using 3.309 to substitute for $(T_0 - T_\infty)$ in equation 3.307, we obtain,

$$T - T_\infty = \frac{Q}{4\pi K r} \tag{3.310}$$

When expressed in this manner, the temperature field does not depend on the radius of the sphere R, but only on the total heat emitted by the sphere per unit time Q. Therefore, this solution is valid outside a sphere of any radius, provided the total heat emitted per unit time is Q. Specifically, it is also valid in the limit $R \to 0$, the 'point source'. The temperature field due to a point source will be an important concept in our later discussion of the harmonic expansions.

3.6.3 Unsteady conduction in a sphere:

This problem is the analogue, in spherical co-ordinates, of the unsteady conduction into a cylinder. A spherical particle of radius R and initial temperature T_0 is immersed into a fluid with temperature T_∞. The volume of the surrounding fluid is considered to be large enough that the heat generated from the particle does not increase the temperature of the fluid. We would like to find out the variation of the temperature in the particle with time.

In the long time limit, we would expect the temperature of the particle to be equal to T_∞. Therefore, we can define a scaled temperature $T^* = (T - T_\infty)/(T_0 - T_\infty)$, so that the scaled temperature is zero everywhere in the long time limit. The scaled radius is defined as $r^* = (r/R)$, and the scaled time is $t^* = (t\alpha/R^2)$. With this, the unsteady energy balance equation becomes,

$$\frac{\partial T^*}{\partial t^*} = \frac{1}{r^{*2}} \frac{\partial}{\partial r^*} \left(r^{*2} \frac{\partial T^*}{\partial r^*} \right) \tag{3.311}$$

with the boundary condition at the surface of the sphere,

$$T^* = 0 \quad \text{at} \quad r^* = 1 \tag{3.312}$$

and initial condition that the temperature is equal to T_0 everywhere within the sphere at $t = 0$,

$$T^* = 1 \quad \text{at}$$

t$^* = 0$ for $r^* < 1$(3.313)

Equation 3.311 is a second order differential equation in the radial co-ordinate r^*, but we have specified only one boundary condition 3.312 at the surface of the sphere. The second condition is the 'symmetry' condition at $r^* = 0$, similar to the condition 3.205 at the axis of the cylindrical co-ordinate system. Since the temperature field is spherically symmetric, the derivative of the temperature at the center will be independent of the direction of approach only if it is zero at the center,

$$\frac{\partial T^*}{\partial r^*} = 0 \quad \text{at} \quad r^* = 0 \tag{3.314}$$

This is the symmetry condition at the origin of the spherical co-ordinate system.

Equation 3.311 is solved using the separation of variables procedure, where the temperature field is written as,

$$T^*(r^*, t^*) = \mathcal{R}(r^*)\mathcal{T}(t^*) \tag{3.315}$$

Using the separation of variables procedure (substitute the above expression for T^* into the conservation equation 3.311, and divide the equation by T^*), we obtain equations for $\mathcal{R}$ and $\mathcal{T}$,

$$\frac{1}{\mathcal{R}}\frac{1}{r^{*2}}\frac{\partial}{\partial r^*}\left(r^{*2}\frac{\partial \mathcal{R}}{\partial}\right) = -\beta^2 \tag{3.316}$$

$$\frac{1}{\mathcal{T}}\frac{\partial \mathcal{T}}{\partial t^*} = -\beta^2 \tag{3.317}$$

where β is the eigenvalue to be determined from the boundary condition at the surface of the sphere.

The equation 3.316 for $\mathcal{R}$ can be simplified as,

$$r^{*2}\frac{\partial^2 \mathcal{R}}{\partial r^{*2}} + 2r^*\frac{\partial \mathcal{R}}{\partial r^*} + \beta^2 r^{*2}\mathcal{R} = 0 \tag{3.318}$$

The general solution for this equation is,

$$\mathcal{R}(r^*) = \frac{C \sin(\beta r^*)}{r^*} + \frac{D \cos(\beta r^*)}{r^*} \tag{3.319}$$

where C and D are constants to be determined from the boundary conditions. From the symmetry boundary condition 3.314, it is clear that C has to be equal to zero. The constant β is determined from the boundary condition 3.312, which is equivalent to $\mathcal{R} = 0$ at $r^* = 1$. Therefore, the eigenvalue β assumes discrete values,

$$\beta_n = n\pi \tag{3.320}$$

where n is an integer. The solution of equation 3.317 for $\mathcal{T}(t^*)$ is,

$$\mathcal{T}(t^*) = \exp(-\beta_n^2 t^*) = \exp(-n^2\pi^2 t^*) \tag{3.321}$$

Therefore, the general solution for the temperature field is,

$$\begin{aligned} T^* &= \sum_{n=1}^{\infty} C_n r^{*-1} \sin(n\pi r^*) \exp(-n^2\pi^2 t^*) \\ &= \sum_{n=1}^{\infty} C_n \Psi_n(r^*) \exp(-n^2\pi^2 t^*) \end{aligned} \tag{3.322}$$

where $\Psi_n = (\sin(n\pi r^*)/r^*)$ is the set of basis functions for the differential equation 3.318.

The constants C_n are determined from the initial condition,

$$T^* = 1 \quad \text{at} \quad t^* = 0 \tag{3.323}$$

which is equivalent to,

$$\sum_{n=1}^{\infty} \frac{C_n \sin(n\pi r^*)}{r^*} = 1 \tag{3.324}$$

The orthogonality relations for the basis functions $\Psi_n = (\sin(n\pi r^*)/r^*)$ for the present problem,

$$\begin{aligned} \langle \Psi_n, \Psi_m \rangle &= \int_0^1 r^{*2} dr^* \; \frac{\sin(n\pi r^*)}{r^*} \frac{\sin(m\pi r^*)}{r^*} \\ &= \frac{\delta_{mn}}{2} \end{aligned} \tag{3.325}$$

where δ_{mn} is 1 for $m = n$, and 0 for $m \neq n$. Using this, we obtain the solution for the constants C_n,

$$C_n = 2 \tag{3.326}$$

3.6.4 Similarity solution for diffusion from a point source:

The scaled temperature is defined as $T^* = ((T - T_\infty)/T_\infty)$, so that the scaled temperature is zero in the limit $r \to \infty$. The heat conduction equation in the fluid is,

$$\frac{\partial T^*}{\partial t} = \alpha \left(\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dT^*}{dr} \right) \right) \tag{3.327}$$

One of the boundary conditions are that $T^* \to 0$ in the limit $r \to \infty$, while the other is the flux condition at the source, which requires that the total amount of heat emitted per unit time is Q. Therefore, the flux from a spherical surface of radius r around the point source is $(Q/(4\pi r^2))$

$$-KT_\infty \frac{dT^*}{dr} = \frac{Q}{4\pi r^2} \tag{3.328}$$

in the limit $r \to 0$.

As before, the equation **??** can be expressed in terms of a similarity variable,

$$\frac{d^2T^*}{d\xi^2} + \left(\frac{2}{\xi} + \frac{\xi}{2} \right) \frac{dT^*}{d\xi} = 0 \tag{3.329}$$

This equation can be solved to obtain the

$$\frac{dT^*}{d\xi} = \frac{C}{\xi^2} \exp{(-\xi^2/4)} \tag{3.330}$$

The temperature can be obtained by integrating the above equation with respect to ξ, and realising that $T = 0$ as $\xi \to \infty$.

$$T^* = \int_\infty^\xi d\xi' \frac{C}{\xi'^2 \exp{(-\xi'^2/4)}(3.331)}$$

The above equation shows that the temperature at the surface of the wire is undefined, because the integral in equation **??** increases proportional to ξ^{-1} as $\xi \to \infty$.

The constant C can be determined from the flux condition, in the limit $r \to 0 (\xi \to 0)$,

$$K \frac{dT^*}{dr} = -\frac{Q}{4\pi r^2} \tag{3.332}$$

When expressed in terms of ξ, this is equivalent to

$$\frac{dT^*}{d\xi} = -\frac{Q}{2\pi K \xi} \tag{3.333}$$

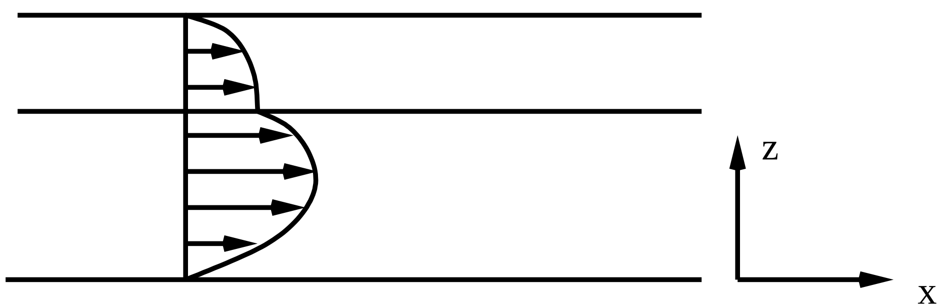


Figure 3.15: Flow of immiscible fluids.

Inserting solution **??** into the above equation, we find that the constant $C = (Q/2\pi K)$. Therefore, the solution for the temperature field is,

$$T^* = \frac{Q}{2\pi k}\int_{\infty}^{\xi} d\xi' \frac{1}{\xi'} \exp\left(-\xi'^2/4\right) \tag{3.334}$$

Adjacent flow of two immiscible liquids.

This is an example which illustrates the application of boundary conditions between two immiscible liquids. Two immiscible liquids flow through a channel of length L and width W under the influence of a pressure gradient, as shown in figure 3.15. The flow rates are adjusted such that the channel is half filled with fluid I (more dense phase) and half filled with fluid II (less dense phase). It is necessary to find the distribution or velocity in this case.

The momentum balance is similar to that for the flow in a pressure gradient shown in the last example. The pressure balance reduces to

$$\frac{d\tau_{xz}}{dx} + \frac{p_0 - p_L}{L} = 0 \tag{3.335}$$

This equation is valid in either phase I or phase II. Integration gives two relations in the two phases.

$$\begin{aligned} \tau_{xz}^{(I)} &= -\left(\frac{p_0 - p_L}{L}\right) x + C^{(I)} \\ \tau_{xz}^{(II)} &= -\left(\frac{p_0 - p_L}{L}\right) x + C^{(II)} \end{aligned} \tag{3.336}$$

We can make use of one of the boundary conditions, that the stress is equal at the interface, to relate $C^{(I)}$ and $C^{(II)}$.

$$\begin{aligned} \text{At} x = 0 \qquad & \tau_{xz}^{(I)} = \tau_{xz}^{(II)} \\ & C^{(I)} = C^{(II)} \end{aligned} \tag{3.337}$$

Using Newton's law of viscosity to relate the stress to the strain rate, we get

$$\begin{aligned} \mu^{(I)} \frac{dv_z^{(I)}}{dx} &= -\left(\frac{p_0 - p_L}{L}\right) x + C^{(I)} \\ \mu^{(II)} \frac{dv_z^{(II)}}{dx} &= -\left(\frac{p_0 - p_L}{L}\right) x + C^{(II)} \end{aligned} \tag{3.338}$$

This can be integrted to give

$$\begin{aligned} v_z^{(I)} &= -\left(\frac{p_0 - p_L}{2\mu^{(I)} L}\right) x^2 + \frac{C^{(I)} x}{\mu^{(I)}} + C_2^{(I)} \\ v_z^{(II)} &= -\left(\frac{p_0 - p_L}{2\mu^{(II)} L}\right) x^2 + \frac{C^{(I)} x}{\mu^{(I)}} + C_2^{(II)} \end{aligned} \tag{3.339}$$

There are three constants in the above equations, which are determined using the three available boundary conditions.

$$\begin{aligned} \text{At} x = 0 \qquad & v_z^{(I)} = v_z^{(II)} \\ \text{At} x = -b \qquad & v_z^{(I)} = 0 \\ \text{At} x = b \qquad & v_z^{(II)} = 0 \end{aligned} \tag{3.340}$$

These boundary conditions provide three relationships between the constants $C^{(I)}$, $C_2^{(I)}$ and $C_2^{(II)}$.

$$\begin{aligned} C_2^{(I)} &= C_2^{(II)} \\ 0 &= -\left(\frac{p_0 - p_L}{2\mu^{(I)} L}\right) b^2 - \frac{C^{(I)} b}{\mu^{(I)}} + C_2^{(I)} \\ 0 &= -\left(\frac{p_0 - p_L}{2\mu^{(II)} L}\right) b^2 + \frac{C^{(I)} b}{\mu^{(I)}} + C_2^{(I)} \end{aligned} \tag{3.341}$$

These can be solved to obtain

$$\begin{aligned} C_1 &= \frac{(p_0 - p_L) b}{2L} \left(\frac{\mu^{(I)} - \mu^{(II)}}{\mu^{(I)} + \mu^{(II)}}\right) \\ C_2^{(I)} &= \frac{(p_0 - p_L) b^2}{2\mu^{(I)} L} \left(2 \frac{\mu^{(I)}}{\mu^{(I)} + \mu^{(II)}}\right) \end{aligned} \tag{3.342}$$

Therefore, the velocity profiles in each layer are

$$\begin{aligned}
v_z^{(I)} &= \frac{(p_0 - p_L)b^2}{2\mu^{(I)}L}\left[\left(\frac{2\mu^{(I)}}{\mu^{(I)}+\mu^{(II)}}\right) + \left(\frac{\mu^{(I)}-\mu^{(II)}}{\mu^{(I)}+\mu^{(II)}}\right)\frac{x}{b} - \frac{x^2}{b^2}\right] \\
v_z^{(I)} &= \frac{(p_0 - p_L)b^2}{2\mu^{(II)}L}\left[\left(\frac{2\mu^{(II)}}{\mu^{(I)}+\mu^{(II)}}\right) + \left(\frac{\mu^{(I)}-\mu^{(II)}}{\mu^{(I)}+\mu^{(II)}}\right)\frac{x}{b} - \frac{x^2}{b^2}\right] \quad (3.343)
\end{aligned}$$

Exercises

1. Consider a long and narrow channel two - dimensional of length L and height H, where $H \ll L$. The ends of the channel are closed so that no fluid can enter or leave the channel. The bottom and side walls of the channel are stationary, while the top wall moves with a velocity $V(t)$. Since the length of the channel is large compared to the height, the flow near the center can be considered as one dimensional. Near the ends, there will be some circulation due to the presence of the side walls, but this can be neglected far from the sides. For the flow far from the walls of the channel,

   (a) Write the equations for the unidirectional flow. What are the boundary conditions? What restriction is placed on the velocity profile due to the fact that the ends are closed and fluid cannot enter or leave the channel?

   (b) If the wall is given a steady velocity V which is independent of time, solve the equations (neglecting the time derivative term). Calculate the gradient of the pressure.

   (c) If the wall is given an oscillating velocity $V\cos(\omega t)$, obtain an ordinary differential equation to obtain the velocity profile. Get an analytical solution for this which involves the constants of itegration. Use the boundary conditions to determine all unknown constants.

2. In shell-and-tube heat exchangers, the tube side often has fins in order to increase the conduction rate, as shown in figure 1. The fin can be modeled, in two dimensions, as a rectangular block of length L, height H and with thermal conductivity k. One surface (outer wall of the tube of the heat exchanger) is at the temperature T_t, which is the temperature of the tube side fluid. The other three surfaces are

at the temperature T_s, which is the temperature of the shell side fluid. Determine the heat flux from the fin as follows.

(a) Write down the conduction equation, $\nabla^2 T = 0$, in two dimensions, and specify the boundary conditions.

(b) Define a non-dimensional temperature in such a way that both boundary conditions are homogeneous along one of the co-ordinates.

(c) Use separation of variables to obtain separate the dependence of T on the x and y co-ordinates.

(d) Write down the final solution for the temperature field which satisfies homogeneous boundary conditions.

(e) Determine the coefficients using orthogonality relations along the inhomogeneous direction. From this, calculate the heat flux as a function of the temperature difference.

3. A rectangular channel of width W and height H is used for transporting fluid of density ρ and viscosity μ. If a steady pressure difference Δp is applied across the length L of the channel, determine the flow rate.

   (a) First, obtain the momentum balance for the streamwise velocity for a differential volume in the channel.

   (b) Solve the equation using separation of variables to obtain the velocity. Note that the fluid velocity is zero on all the walls of the channnel.

4. Consider a cylinder in a thin annular region, as shown in figure 3.16. The cylinder is pulled with a constant velocity V. The pressure is equal on both sides of the cylinder. Determine the fluid velocity, and the flow rate.

5. A fluid is contained in the annular region between two concentric cylinders of radius R_1 and R_2 moving with angular velocities Ω_1 and Ω_2. The gravitational field acts along the axis of the cylinders as shown in figure. The vessel is tall enough that the flow can be considered unidirectional when the distance from the bottom is large compared to the gap width $(R_1 - R_2)$. In this case, choose a coordinate system and write down the mass and momentum conservation equations. Solve these for

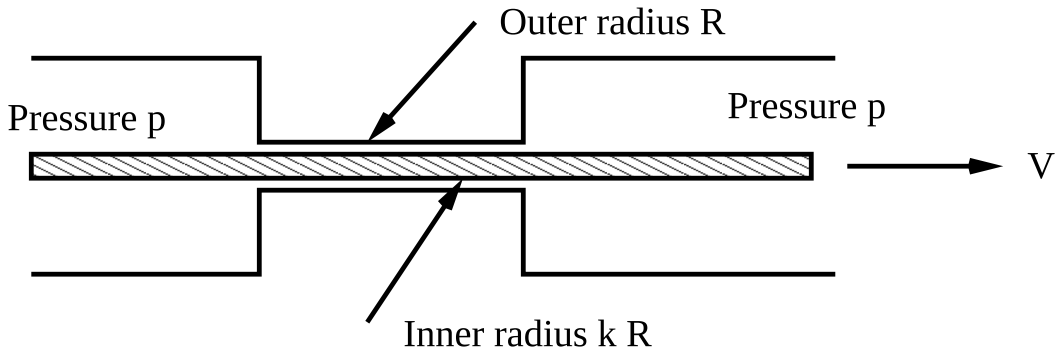


Figure 3.16: Wire coating of dies.

the pressure and velocity fields. Can you find the equation for the free surface?

6. Consider the unidirectional flow of a 'power law fluid' through a pipe of radius R, where the flow takes place only in the axial direction and there is no radial flow. Use a polar coordinate system where the flow is in the x direction, and the radial vector is in the r direction. The constitutive relation for the shear stress τ_{xr} for this fluid is given by

$$\tau_{xr} = k_n \left| \frac{du_x}{dr} \right|^{n-1} \left(\frac{du_x}{dr} \right)$$

   (a) Use a momentum balance in the axial direction to determine equation for the variation in the velocity as a function of the pressure gradient.

   (b) What are the boundary conditions?

   (c) Assume the pressure gradient is a constant, and solve the equation to obtain the velocity profile.

7. Consider the annular flow of two fluids in a tube, as shown in figure 2, driven by a pressure difference applied across the ends of the tube. The tube radius is R_o, while the radius of the interface between the two fluids is R_i. The viscosities of the two fluids are μ_i and μ_o. Assume that the densities of the two fluids are equal.

(a) Write down momentum balance equations for the two fluids. (Do not try to derive it, but use expressions already derived in the notes).

(b) What are the boundary conditions, and the interface conditions between the two fluids?

(c) How are the pressures and pressure gradients in the two fluids related?

(d) Solve the momentum conservation equations to obtain the velocity fields in the two fluids. From this, calculate the relationship between friction factor and the Reynolds numbers based on the average velocity and viscosity of the two fluids.

8. A resistance heating appratus for a fluid consists of a thin wire immersed in a fluid. In order to design the appratus, it is necessary to determine the temperature in the fluid as a function of he heat flux from the wire. For the purposes of the calculation, the wire can be considered of infinite length so that the heat conduction problem is effectively a two dimensional problem. In addition, the thickness of the wire is considered small compared to any other length scales in the problem, so that the wire is a line source of heat. The wire and the fluid are initially at a temperature T_0. At time $t = 0$, the current is switched on so that the wire acts as a source of heat, and the heat transmitted per unit length of the wire is Q. The heat conduction in the fluid is determined by the unsteady state heat conduction equation

$$\partial_t T = K \nabla^2 T \tag{3.344}$$

and the heat flux (heat conducted per unit area) is

$$K \nabla T \tag{3.345}$$

where K is the thermal conductivity of the fluid.

(a) Choose an appropriate coordinate system, and write down the unsteady heat conduction equation.

(b) What are the boundary conditions? Give special attention o the heat flux condition at the wire, and note that the wire is considered to be of infinitesimal radius.

(c) Solve the heat conduction equation using the simplest method, and determine the temperature field in the fluid.

(d) Use the boundary conditions to determine the constants in the expression for the temperature field.

9. Consider a *two dimensional* incompressible flow in a diverging channel with subtended angle α bounded by solid walls as shown in the figure below. The flux of fluid through the channel is Q. The two dimensional (r, ϕ) polar coordinate system used for the analysis is also shown in the figure. Assume a one dimensional velocity field $u_r \neq 0$ and $u_\phi = 0$. For this case,

   (a) From the mass conservation equation, determine the form of the velocity in the radial direction.

   (b) Write down the momentum conservation equations in the r and θ directions and simplify assuming that the Reynolds number is small, so that inertial terms can be neglected. Simplify the equations. What are the boundary conditions?

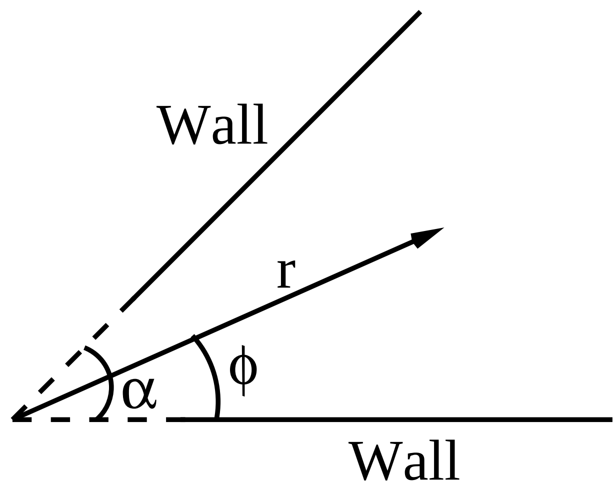
   (c) Eliminate the pressure from the momentum equations to obtain an equation for the velocity field.

   (d) Solve the equation to obtain an expression for the velocity. How many constants are there, and how are they determined? *Do not determine the constants.*

   (e) If we assume that the Reynolds number is high, so that potential flow conditions apply, what are the governing equations and boundary conditions?

   (f) What are the velocity and pressure fields in this case?

10. Consider an annular channel described in a cylindrical (r, θ, z) coordinate system. The cross section of the channel is shown in figure 1, and the z direction is perpendicular to the plane of the cross section. The channel is bounded by solid walls at $r = 0$ and $r = R$, and at $\theta = 0$ and $\theta = \Theta$. The wall at $r = R$ is moving in the z direction with a velocity U, while those at $\theta = 0$ and $\theta = \Theta$ are stationary. The flow is a unidirectional, fully developed, and steady flow with velocity only



in the z direction. The equation for the velocity field in the z direction is,

$$\mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} \right) = 0$$

where μ is the viscosity.

(a) What are the boundary conditions required for solving the above equation?

(b) Use separation of variables, by writing $u_z = F(r)T(\theta)$, and obtain equations for F and T.

(c) Find the solution for $T(\theta)$ that satisfies the boundary conditions.

(d) Find the solution for $F(r)$, and enforce boundary conditions to find the final solution.

11. A piston damper assembly, shown in figure 1, consists of a cylindrical piston rod moving through a stationary cylindrical sleeve which is filled with fluid. The sleeve is closed at both ends so that the fluid cannot move in or out of the sleeve. The radius of the piston rod is R_p, while the outer radius of the sleeve is R_s. The length of the sleeve L is large compared to the radius R_s. The piston rod moves with a velocity U with respect to the sleeve. Consider the region away from the ends of the sleeve, where the flow is expected to be in only one direction (parallel to the walls).

Figure 3.17:

(a) Which is the non-zero component of the velocity in this co-ordinate system, and which spatial co-ordinate does the velocity depend on? Write the boundary conditions in the center region away from the ends of the sleeve, where the flow is expected to be unidirectional.

(b) What condition can be obtained about the velocity in the center of the channel (away from the ends) from mass conservation?

(c) Use a shell balance to derive the momentum conservation equation for the unidirectional flow in the center of the sleeve away from the ends.

(d) Solve the momentum conservation equation to determine the velocity field.

12. Consider the fully developed flow in a circular tube with velocity profile

$$u_x = U(t)\left(1 - \frac{r^2}{R^2}\right)$$

as shown in figure 1, where the maximum velocity U could be a function of t, but is independent of the stream-wise co-ordinate x. There is

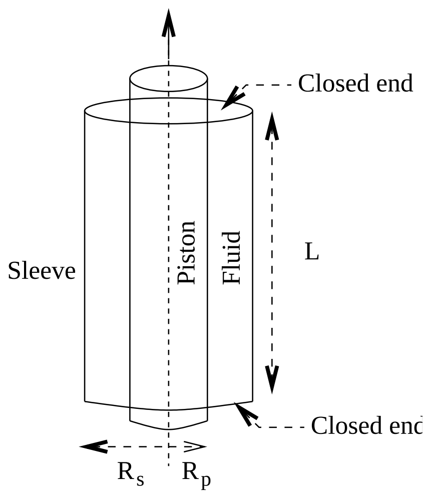


Figure 3.18:

viscous dissipation which generates heat within the fluid, and the heat generated per unit volume of the fluid per unit time is given by,

$$Q = \mu \left(\frac{\partial u_x}{\partial r} \right)^2$$

Due to this, there is a temperature variation across the tube, and the temperature field is governed by the convection-diffusion equation,

$$\rho C_v \left(\frac{\partial T}{\partial t} + \mathbf{u}.\nabla T \right) = k \nabla^2 T + Q$$

The temperature at the wall of the tube is maintained at $T = T_0$, and **we assume that the temperature could be a function of time, but the temperature field is 'fully developed' so that the temperature is independent of the flow (x) direction**.

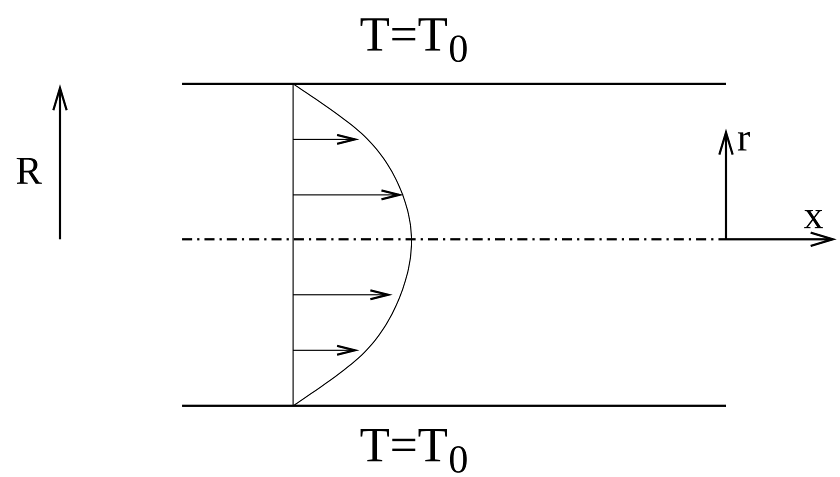
(a) Choose a suitable co-ordinate system, and write down the convection-diffusion equation for the time-dependent but 'fully developed' temperature field.

(b) Scale the co-ordinates and time. What would you use to scale the temperature?

(c) Obtain the solution for the temperature at steady state, where both the maximum velocity U and temperature are independent of time.

(d) If the maximum velocity has a sinusoidal variation, $U(t) = U \cos(\omega t)$, what is the value of the heat source Q? How would you express the inhomogeneous term in the time-dependent convection-diffusion equation for the temperature in order to obtain a solution?

(e) Determine the solution for the temperature field.



13. An ideal vortex is a flow with circular streamlines where the particle motion is incompressible and irrotational. The velocity profile obeys the equation in cylindrical coordinates:

$$v_\theta = \frac{\Gamma}{2\pi r} \tag{3.346}$$

with $v_r = v_z = 0$. At the origin, the above equation indicates that the velocity becomes infinite. But this is prohibited because viscous forces become important and the flow is rotational in a small region near the core.

Consider an ideal vortex in which the velocity is given by the above equation for $t < 0$, and the core velocity is constrained to be zero at $t = 0$. Find the velocity profile for $t > 0$. Assume that v_θ is the only non - zero velocity component.

14. A cubic solid of side a is initially held at a temperature T_0. At times $t \geq 0$, its lateral faces are held at temperatures T_A, T_B, T_0 and T_0 as illustrated in figure 3.19. The top and bottom faces are insulated so that no heat is transferred through them. The cube has heat conductivity C, density ρ and thermal conductivity K.

    (a) Solve for the steady state temperature in the cube.

    (b) Show how the transient problem may be set up in a form to which separation of variables can be applied.

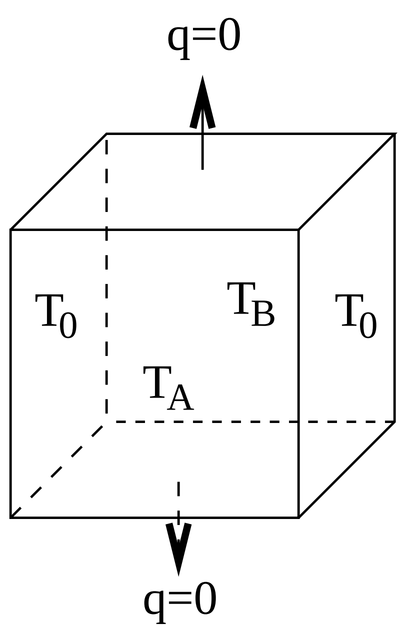


Figure 3.19: Conduction from a cube.

15. A rotating cylinder geometry consists of a cylinder of radius R and height H, filled with fluid, with two end caps. The cylinder rotates with an angular velocity Ω, while the end caps are stationary. Determine the fluid velocity field using separation of variables as follows.

    (a) Choose a coordinate system for the problem. Clearly, the only non-zero component of the velocity is u_ϕ. Determine the boundary conditions for this component of the velocity.

(b) Write down the mass balance condition for an incompressible fluid. For a uni-directional flow in which the density is a constant, what does this reduce to?

(c) Use a shell balance to determine the conservation equation for the velocity. Can you eliminate the pressure term using a result from the mass balance condition?

(d) Solve the conservation equation at steady state using the method of separation of variables. Frame the orthogonality conditions which would be required to solve the problem.

Data:

(a) Bessel equation:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

Solution:

$$y = A_1 J_n(x) + A_2 Y_n(x)$$

where $J_n(x)$ is bounded for $x \to 0$, and $Y_n(x)$ is bounded for $x \to \infty$. tem Modified Bessel equation:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + n^2)y = 0$$

Solution:

$$y = A_1 I_n(x) + A_2 K_n(x)$$

where $I_n(x)$ is bounded for $x \to 0$, and $K_n(x)$ is bounded for $x \to \infty$.