Chapter 5

Steady and unsteady diffusion

In this chapter, we solve the diffusion and forced convection equations, in which it is necessary to evaluate the temperature or concentration fields when the velocity field is known.

5.1 Spherical coordinates

The solution of the equation:

$$\partial_t c = D\nabla^2 c \tag{5.1}$$

in spherical coordinates can be obtained using the separation of variables. In spherical coordinates, 5.1 is:

$$\partial_t c = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial c}{\partial\theta}\right) + \frac{1}{r^2\sin(\theta)^2}\frac{\partial^2 c}{\partial\phi^2} \tag{5.2}$$

where T is a general function of (r, θ, ϕ, t). In the separation of variables technique, the variable T is written as:

$$c = F(t)R(r)\Theta(\theta)\Phi(\phi) \tag{5.3}$$

Inserting this into $\mathring{4}215$, and dividing the entire equation by $F(t)R(r)\Theta(\theta)\Phi(\phi)$, we get:

$$\frac{1}{D}\frac{1}{F(t)}\frac{dF(t)}{dt} = \frac{1}{R}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{r^2\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{1}{r^2\sin(\theta)^2}\frac{d^2\Phi}{d\phi^2} \tag{5.4}$$

Note that the partial derivatives have now become total derivatives because F is only a function of t, and R, Θ and Φ are functions of only r, θ and ϕ only. The left side of the equation is only a function of t, while the right side is only a function of (r, θ, ϕ). Therefore, each side has to be equation to a constant, say $-\lambda^2$. The solution for F can be easily obtained:

$$F(t) = \exp\left(-\lambda^2 t\right) \tag{5.5}$$

Note that it is necessary for the constant $-\lambda^2$ to be negative for the solution to be bounded in the limit $t \to \infty$.

The remainder of the equation 5.4 can now be written as:

$$\sin(\theta)^2 \left[\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) + \lambda^2 r^2\right] = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \tag{5.6}$$

Here, the left side is only a function of r and θ, and the right side is only a function of ϕ. Therefore, both these have to be equal to a constant, say m^2. This can be easily solved for the function $\Phi(\phi)$:

$$\Phi = A_1 \sin(m\phi) + A_2 \cos(m\phi) \tag{5.7}$$

Note that m has to be an integer, because the physical system obtained remains the same when ϕ is increased by an angle 2π.

The rest of equation 5.6 can now be written as:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \lambda^2 r^2 = -\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) + \frac{m^2}{\sin(\theta)^2} \tag{5.8}$$

Here, the right side is only a function of θ, while the left side is only a function of r. Therefore, both sides can be set equal to a constant, $n(n+1)$. The equation for Θ then becomes

$$-\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) + \frac{m^2}{\sin(\theta)^2} = n(n+1) \tag{5.9}$$

The solutions of the above equation are called 'associated Legendre functions of degree n and order m':

$$\Theta(\theta) = B_1 P_n^m(\cos(\theta) + B_2 Q_n^m(\cos(\theta) \tag{5.10}$$

Note that just as m was constrained to be an integer in 5.7 and $\mathring{4}221$ because the solution is required to remain unchanged when ϕ is increased by 2π, in the above equation n is also an integer because of the functions with non - integer values of n become infinite at $\theta = 0$. In addition, there is another stipulation that $n > m$, because Θ diverges at $\theta = 0$ for $n < m$. Further, the functions $Q_n^m(\cos(\theta))$ also become infinite for $\theta = \pi$, and therefore the only solution for Θ which is finite for both $\theta = 0$ and $\theta = \pi$ is:

$$\Theta = B_1 P_n^m(\cos(\theta)) \tag{5.11}$$

The above solutions are called 'Legendre polynomials', because the series solution for the Legendre equation 5.11 terminates at a certain order. The first few polynomials are:

$$\begin{array}{llll} P_0^0 & = 1 & P_1^0(x) & = x \\ P_2^0(x) & = \frac{1}{2}(3x^2 - 1) & P_1^1(x) & = -\sqrt{1-x^2} \\ P_2^1(x) & = -3x\sqrt{1-x^2} & P_2^2(x) & = 3(1-x^2) \end{array} \tag{5.12}$$

Finally, the function $R(r)$ can be determined by solving the left side of 5.8:

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[\lambda^2 - \frac{n(n+1)}{r^2}\right] R = 0 \tag{5.13}$$

The solutions of this equation are called 'spherical Bessel functions', which are given by:

$$R(r) = C_1 j_n(\lambda r) + C_2 y_n(\lambda r) \tag{5.14}$$

where the functions j_n and y_n are related to Bessel functions:

$$\begin{aligned} j_n(x) &= \sqrt{\pi/2x} J_{n+\frac{1}{2}}(x) \\ y_n(x) &= \sqrt{\pi/2x} Y_{n+\frac{1}{2}}(x) \end{aligned} \tag{5.15}$$

In case we are considering a system at steady state, where the time derivative of T is zero (so that λ is zero), the spherical Bessel functions assume relatively simple forms:

$$R(r) = C_1 r^n + C_2 r^{-n-1} \tag{5.16}$$

Having obtained the above solutions, we can derive a general solution for T:

$$\begin{aligned} T &= R(r)\Theta(\theta)\Phi(\phi)F(t) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n-1} [C_{1n} j_n(\lambda r) + C_{2n} y_n(\lambda r)] P_n^m(\cos(\theta)) \\ &\qquad \times [A_{1m}\sin(m\phi) + A_{2m}\cos(m\phi)] \exp(-\lambda^2 t) \end{aligned} \tag{5.17}$$

where the functions C_{1n}, C_{2n}, A_{1m} and A_{2n} are to be determined from the boundary conditions.

The solutions 4̊220 and 4̊224 are eigenfunctions of Sturm - Liouville equations, and therefore the eigenfunctions are complete and orthogonal. This orthogonality can be used to determine the constants C_{1n}, C_{2n}, A_{1n} and A_{2n} in 4̊230. The orthogonality of the sine and cosine functions are expressed as:

$$\begin{aligned}
\int_0^{2\pi} d\phi\ \sin(m\phi)\sin(m'\phi) &= \left[\begin{array}{ll} 0 & \text{for} \quad m \neq m' \\ \dfrac{\pi}{2} & \text{for} \quad m = m' \end{array}\right. \\
\int_0^{2\pi} d\phi\ \cos(m\phi)\cos(m'\phi) &= \left[\begin{array}{ll} 0 & \text{for} \quad m \neq m' \\ \dfrac{\pi}{2} & \text{for} \quad m = m' \end{array}\right. \\
\int_0^{2\pi} d\phi\ \sin(m\phi)\cos(m'\phi) &= \left[\begin{array}{llll} 0 & \text{for} \quad m \neq n & \text{and} \quad (m+n) & \text{even} \\ \dfrac{2m}{m^2-n^2} & \text{for} \quad m \neq n & \text{and} \quad (m+n) & \text{odd} \\ 0 & \text{for} \quad m = n & & \end{array}\right.
\end{aligned} \tag{5.18}$$

Similarly, the Legendre polynomials also satisfy the following orthogonality constraints in the interval $0 \leq \theta \leq \pi$:

$$\int_0^{\pi} d\theta\ P_n^m(\cos(\theta))P_{n'}^m(\cos(\theta)) = \left[\begin{array}{ll} 0 & \text{for} \quad n \neq n' \\ \dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!} & \text{for} \quad n = n' \end{array}\right. \tag{5.19}$$

5.1.1 Diffusion from a sphere

5.1.2 Effective conductivity of a composite

A composite material of thickness L consists of a matrix of thermal conductivity K_m, embedded with particles of radius R and thermal conductivity K_p. A temperature gradient T' is applied across the composite in the x_3 direction, as shown in figure 5.1, and it is necessary to determine the heat flux q_3 as a function of the temperature gradient T' in the x_3 direction. It is assumed that the length L is large compared to the thickness of the particles R, so that the material can be viewed as a continuum with an effective conductivity which is determined by the conduction of heat through the particles and the matrix,

$$q_3 = -K_{eff}T' \tag{5.20}$$

It is necessary to determine K_{eff} as a function of the conductivities of the particles and the matrix.

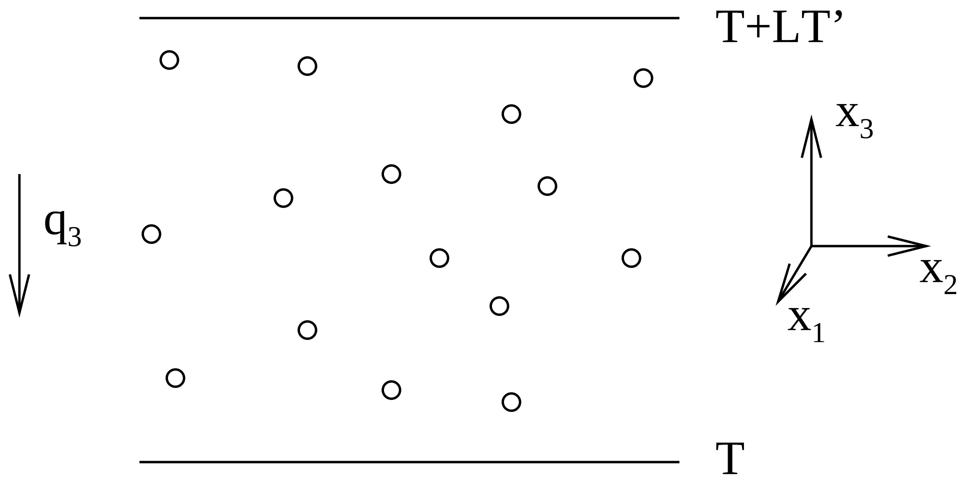


Figure 5.1: Effective conductivity of a composite material.

If there were no particles, the temperature gradient in the material would be uniform, and the conductivity would be equal to the conductivity of the matrix. The presence of the particles causes a disturbance to the temperature field around a particle, which in turn results in a variation in the flux in the vicinity of the particle. If the concentration of particles is small, so that the temperature field around one particle is not influenced by the presence of other particles, the solution can be obtained by considering the effect of an isolated particle in an infinite matrix, and adding this effect for all the particles in the matrix.

In order to solve this problem, we first consider the effect of a particle on the temperature field and heat flux in the matrix. The particle is considered to be located at the origin of the coordinate system, as shown in figure 5.2, and a uniform temperature gradient T' is imposed in the x_3 direction. At steady state, the temperature fields in the particle T_p and in the matrix T_m satisfy the Laplace equation,

$$\begin{aligned} \nabla^2 T_p &= 0 \\ \nabla^2 T_m &= 0 \end{aligned} \tag{5.21}$$

At the interface between the particle and the matrix, the boundary conditions require that the temperature and the heat flux are equal,

$$T_p = T_m$$

$$K_p \frac{\partial T_p}{\partial r} = K_m \frac{\partial T_m}{\partial r} \tag{5.22}$$

Though the temperature field is disturbed by the presence of the particle near the particle surface, it is expected that the temperature field is unaltered at a large distance from the particle $r \to \infty$, where it is given by

$$T = T' x_3 = T' r \cos(\theta) \tag{5.23}$$

The temperature field far from the particle can be expressed in terms of the Legendre polynomial solutions for the Laplace equation as

$$T = T' r P_1^0(\cos(\theta)) \tag{5.24}$$

Since the temperature field is driven by the constant temperature gradient imposed at a large distance from the particle, it is expected that the temperature field near the particle will also be proportional to $P_1^0(\cos(\theta))$, since any solution which contains other Legendre polynomials is orthogonal to $P_1^{(0)}(\cos(\theta))$. Therefore, the solutions for T_p and T_m have to be of the form

$$\begin{aligned} T_p &= A_{p1} r P_1^0(\cos(\theta) + \frac{A_{p2}}{r^2} P_1^0(\cos(\theta)) \\ T_m &= A_{m1} r P_1^0(\cos(\theta) + \frac{A_{m2}}{r^2} P_1^0(\cos(\theta)) \end{aligned} \tag{5.25}$$

The constant A_{p2} is identically zero because T_p is finite at $r = 0$, while $A_{m1} = T'$ in order to satisfy the boundary condition in the limit $r \to \infty$. The constants A_{p1} and A_{m2} are determined from the boundary conditions at the surface of the sphers,

$$\begin{aligned} A_{p1} &= \frac{3T'}{2 + K_R} \\ A_{m2} &= \frac{(1 - K_R) R^3}{2 + K_R} \end{aligned} \tag{5.26}$$

The temperature fileds in the particle and the matrix are

$$\begin{aligned} T_p &= \frac{3T' r}{2 + K_R} P_1^0(\cos(\theta) \\ T_m &= T' r P_1^0(\cos(\theta) + \frac{(1 - K_R) R^3 T'}{(2 + K_R) r^2} P_1^0(\cos(\theta)) \end{aligned} \tag{5.27}$$

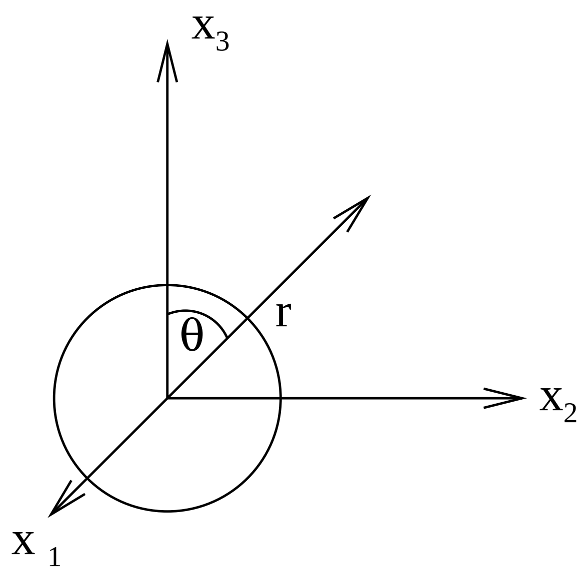


Figure 5.2: Coordinate system for analysing temperature field around the sphere.

The effective conductivity is determined using an effective equation of the form

$$\langle j_3^e \rangle = -K_{eff} T' \tag{5.28}$$

where $\langle j_3^e \rangle$ is the heat flux along the direction of the temperature gradient, which is the direction of the imposed temperature gradient. The average heat flux can be calculated by taking an average over the entire volume of the matrix,

$$\begin{aligned}
\langle j_3^e \rangle &= \frac{1}{V} \int dV j_3^e \\
&= -\frac{1}{V} \left(\int_{\text{matrix}} dV K_m (\nabla T).\mathbf{e}_3 + \int_{\text{particles}} dV K_p (\nabla T).\mathbf{e}_3 \right) \\
&= -\frac{1}{V} \left(\int_{\text{composite}} dV K (\nabla T).\mathbf{e}_3 + \int_{\text{particles}} dV (K_p - K_m)(\nabla T).\mathbf{e}_3 \right) \\
&= -\frac{1}{V} \left(K_m T' + \frac{N V_{\text{particle}}}{V} (K_p - K_m) \frac{3}{2 + K_R} T' \right) \qquad (5.29)
\end{aligned}$$

This gives the effective conductivity

$$K = K_m \left(1 + \frac{3(K_R - 1)}{2 + K_R}\phi\right) \tag{5.30}$$

5.2 Steady diffusion in an infinite domain

5.2.1 Temperature distribution due to a source of energy

Consider a hot sphere of radius R which is continuously generating Q units of energy per unit time into the surrounding fluid, whose temperature at a large distance from the sphere is T_∞, as shown in figure 5.3. The temperature field in the fluid surrounding the sphere can be determined from the energy balance equation at steady state,

$$\nabla^2 T = 0 \tag{5.31}$$

Since the configuration is spherically symmetric, the conservation equation 5.31 is most convenient to solve in a spherical coordinate system in which the origin is located at the center of the sphere. The energy balance equation is then given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0 \tag{5.32}$$

The solution for this equation, which satisfies the condition $T = T_\infty$ at a large distance from the sphere, is

$$T = \frac{A}{r} + T_\infty \tag{5.33}$$

The constant A has to be determined from the condition that the total heat flux at the surface of the sphere is Q,

$$-K\frac{dT}{dr}\bigg|_{r=R}(4\pi R^2) = Q \tag{5.34}$$

This equation can be solved to obtain $A = (Q/4\pi K)$, so that the temperature field is

$$T = \frac{Q}{4\pi K r} + T_\infty \tag{5.35}$$

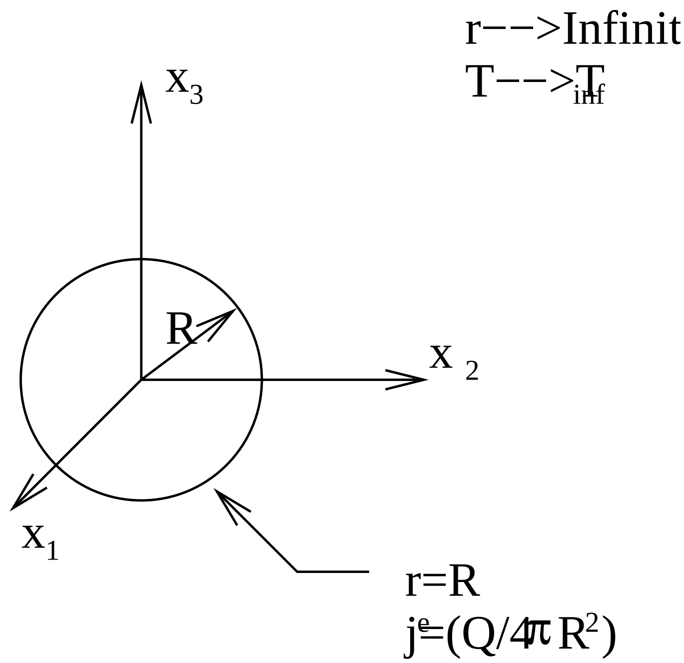


Figure 5.3: Temperature field due to heat generation at the surface of a sphere.

Note that the solution for the temperature, 5.35 does not depend on the radius of the sphere, but only on the total amount of heat generated per unit time from the sphere. If the object that generates the heat were some other irregular object, then the temperature field would depend on the dimensions of the object. However, if we are sufficiently far from the object, so that the distance from the object is large compared to the characteristic length of the object, the object appears as a point source of heat, and the observer cannot discern the detailed shape of the object. In this case, it is expected that the temperature distribution will not depend on the dimensions of the object, but only on the total heat generated per unit time. Mathematically, a point source of energy located at the position x_0 is represented by a delta function,

$$S(\mathbf{x}) = Q\delta(\mathbf{x} - \mathbf{x}_0) \tag{5.36}$$

5.2.2 Dirac delta function

The one-dimensional Dirac delta function, $\delta(x)$, is defined as

$$\delta(x) = 0 \text{ for } x \neq 0 \tag{5.37}$$

$$\int_{-\infty}^{\infty} dx \delta(x) = 1 \tag{5.38}$$

and

$$\int_{-\infty}^{\infty} dx \delta(x) g(x) = g(0) \tag{5.39}$$

It is clear from equation 5.39 that $\delta(x)$ has dimensions of inverse of length. The delta function can be considered the limit of the discontinuous function (figure 5.4)

$$f(x) = (1/h) \text{ for} (-h/2 < x < h/2) \tag{5.40}$$

in the limit $h \to 0$. In this limit, the width of the function tends to zero, while the height becomes infinite, in such a way that the area under the curve remains a constant. It is clear that the function $f(x)$ satisfies all three conditions, 5.37 to 5.39, in the limit $h \to 0$.

In a similar fashion, the three dimensional Dirac delta function, $\delta(x_1, x_2, x_3)$ is defined by

$$\delta(x_1, x_2, x_3) = 0 \text{ for } x_1 \neq 0; x_2 \neq 0; x_3 \neq 0 \tag{5.41}$$

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_3 \delta(x_1, x_2, x_3) = 1 \tag{5.42}$$

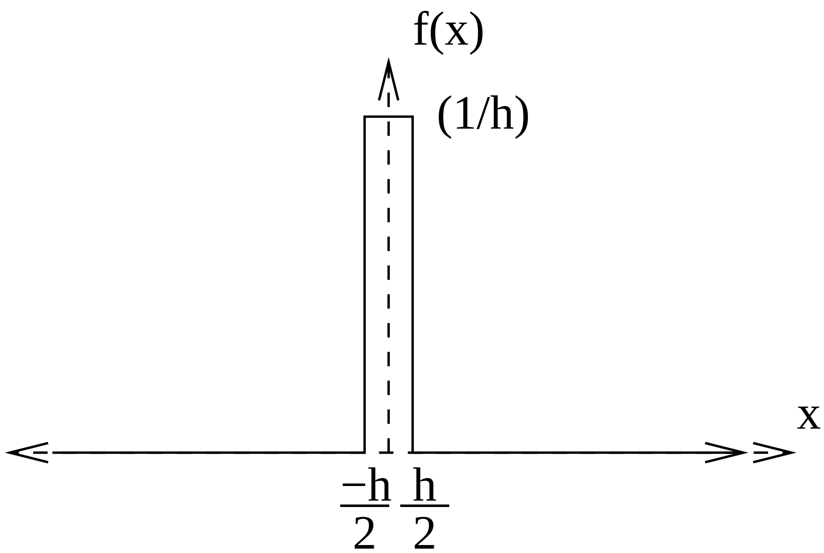


Figure 5.4: Dirac delta function in one dimension.

and

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_3 \delta(x_1, x_2, x_3) g(x_1, x_2, x_3) = g(0, 0, 0) \tag{5.43}$$

The three dimensional Dirac delta function can be considerd as the limit of the function

$$\begin{aligned} f(x_1, x_2, x_3) &= (1/h^3) \text{ for} (-h/2 < x_1 < h/2) \text{and} (-h/2 < x_2 < h/2) \text{and} (-h/2 < x_3 < h/2) \\ &= 0 \text{ otherwise} \end{aligned} \tag{5.44}$$

when $h \to 0$. It is easy to see that this function satisfies all of the above properties.

5.2.3 Temperature distribution due to a point source

The Greens function for an infinite domain is defined as the temperature (or concentration) distribution due to a point source of unit strength.

$$K \nabla^2 G = \delta(\mathbf{x} - \mathbf{x}_0) \tag{5.45}$$

The solution for the Greens function on an infinite domain can be obtained by first shifting the origin of the coordinate system to the position $\mathbf{x}_0$, so that the radius vector $\mathbf{r} = \mathbf{x} - \mathbf{x}_0$, and then solving equation 5.45 in a spherical coordinate system in this coordinate system. In this new coordinate system, the configuration is spherically symmetric, and so the conservation equation 5.45 is

$$K\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial G}{\partial r} = \delta(\mathbf{r}) \tag{5.46}$$

For $\mathbf{r} \neq 0$, the right side of equation 5.46 is equal to zero, so the solution for G (upto an additive constant) is

$$G = \frac{1}{4\pi K r} \tag{5.47}$$

where A is a constant to be determined from the condition at the origin. The condition at the origin is most conveniently determined by integrating equation 5.46 over a small radius ϵ around the origin,

$$-K\int_0^{\epsilon}(4\pi r^2)dr\left(-\frac{A}{r^2}\right) = 1S(\mathbf{x}) = \int d\mathbf{x}'\delta(\mathbf{x}-\mathbf{x}')S(\mathbf{x}') \tag{5.48}$$

The solution for the temperature field is then given by

$$T(\mathbf{x}) = \frac{1}{4\pi K}\int d\mathbf{x}'\frac{S(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}')} \tag{5.49}$$

Example

A wire of length $2L$ immersed in a fluid generates heat at the rate of Q per unit length of the wire per unit time, as shown in figure 5.5. Determine the temperature field due to this wire.

A cylindrical coordinate system is used, where the z axis is along the length of the wire, and the origin is at the center of the wire. The wire is considered to be a line of infinitesimal thickness in the x and y directions, so that the energy source due to the wire per unit length is given by

$$S(\mathbf{x}) = Q\delta(x)\delta(y) \text{ for } -L < z < L \tag{5.50}$$

The temperature field, in terms of this source, is given by

$$\begin{aligned} T(\mathbf{x}) &= \int_0^{\infty} dx' \int_0^{\infty} dy' \int_{-L}^{L} dz' G(x-x', y-y', z-z')S(x', y', z') \\ &= \int_0^{\infty} dx' \int_0^{\infty} dy' \int_{-L}^{L} dz' \frac{Q\delta(x')\delta(y')}{4\pi K}\frac{1}{((x-x')^2+(y-y')^2+(z-z')^2)^{1/2}} \end{aligned}$$

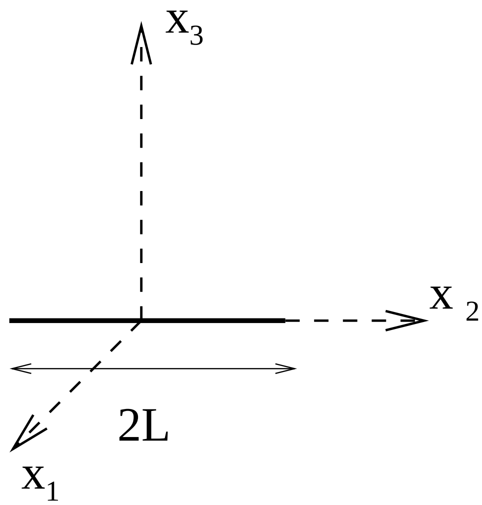


Figure 5.5: Heat generation due to a wire.

$$
\begin{aligned}
&= \frac{Q}{4\pi K}\int_{-L}^{L} dz' \frac{1}{(x^2+y^2+(z-z')^2)^{1/2}} \\
&= \log\left(\frac{L+z+\sqrt{r^2+(L+z)^2}}{-L+z+\sqrt{r^2+(z-L)^2}}\right) \qquad (5.51)
\end{aligned}
$$

The formal solution 5.51 can be more conveniently examined as a function of r at the center of the wire, $z=0$,

$$T(\mathbf{x}) = \log\left(\frac{L+\sqrt{r^2+L^2}}{-L+\sqrt{r^2+L^2}}\right) \qquad (5.52)$$

In the limit $r \gg L$, this solution reduces to

$$T(\mathbf{x}) = \frac{2QL}{4\pi K r} \qquad (5.53)$$

This is the solution for the temperature field due to a point source of energy, as expected when the distance from the source is large compared to the characteristic length of the source. In the opposite limit $r \ll L$, the solution for the temperature field reduces to

$$T(\mathbf{x}) = \frac{Q}{4\pi K}\log\left(\frac{4L^2}{r^2}\right)$$

$$= \frac{Q}{2\pi K}\log\left(\frac{2L}{r}\right) \tag{5.54}$$

We will see, a little later, that this is the temperature field due to an infinite line source of energy in three dimensions, or a point source in two dimensions.

5.2.4 Greens function for finite domains

The Greens function 5.47 is the temperature field due to a point source of unit strength in a fluid of infinite extent. Most practical problems involve finite domains, and it is necessary to obtain a Greens function which satisfies the boundary conditions at the boundaries of the domain. In the case of planar domains, this Greens function is obtained by using 'image charges'. For example, consider a point source located at $\mathbf{x}_s = (L, 0, 0)$ on a semi-infinite domain bounded by a surface at $x_3 = 0$, as shown in figure 5.6, in which the surface has constant temperature equal to T_∞. In this case, the Greens function solution of the type 5.47 does not satisfy the condition $(T - T_{infty}) = 0$ at the surface. However, the boundary condition can be satisfied if we replace the finite domain by an infinite domain, in which there is a source of strength $+1$ at $(L, 0, 0)$, and a source of strength -1 at $\mathbf{x}_I(-L, 0, 0)$, as shown in figure 5.6 (a). It is easily seen that due to symmetry, $(T - T_\infty)$ is equal to zero everywhere on the plane $x_3 = 0$, and the Greens function which satisfies the zero temperature condition is called the Dirichlet Greens function G_D,

$$G_D(\mathbf{x} = \frac{1}{4\pi K|\mathbf{x} - \mathbf{x}_s|} - \frac{1}{4\pi K|\mathbf{x} - \mathbf{x}_I|} \tag{5.55}$$

The diffusion equation is satisfied in the semi-infinite domain $x_3 > 0$ (since the Greens function 5.55 satisfies the diffusion equation), and the boundary conditions are identical to the required boundary conditions at $x_3 = 0$, therefore, the solution G_D is the required solution for the Greens function. Of course, G_D predicts a spurious temperature field in the half plane $x_3 < 0$, but this is outside the physial domain.

A similar Greens function can be obtained if the boundary condition at the surface is a zero normal flux condition, $j_3^e = 0$, at $x_3 = 0$, instead of the zero temperature condition. In this case, the zero flux condition is identically satisfied by imposing a source of equal strength $+1$ at $\mathbf{x}_I = (-L, 0, 0)$, as shown in figure 5.6 (b). The solution for the Greens function with zero flux

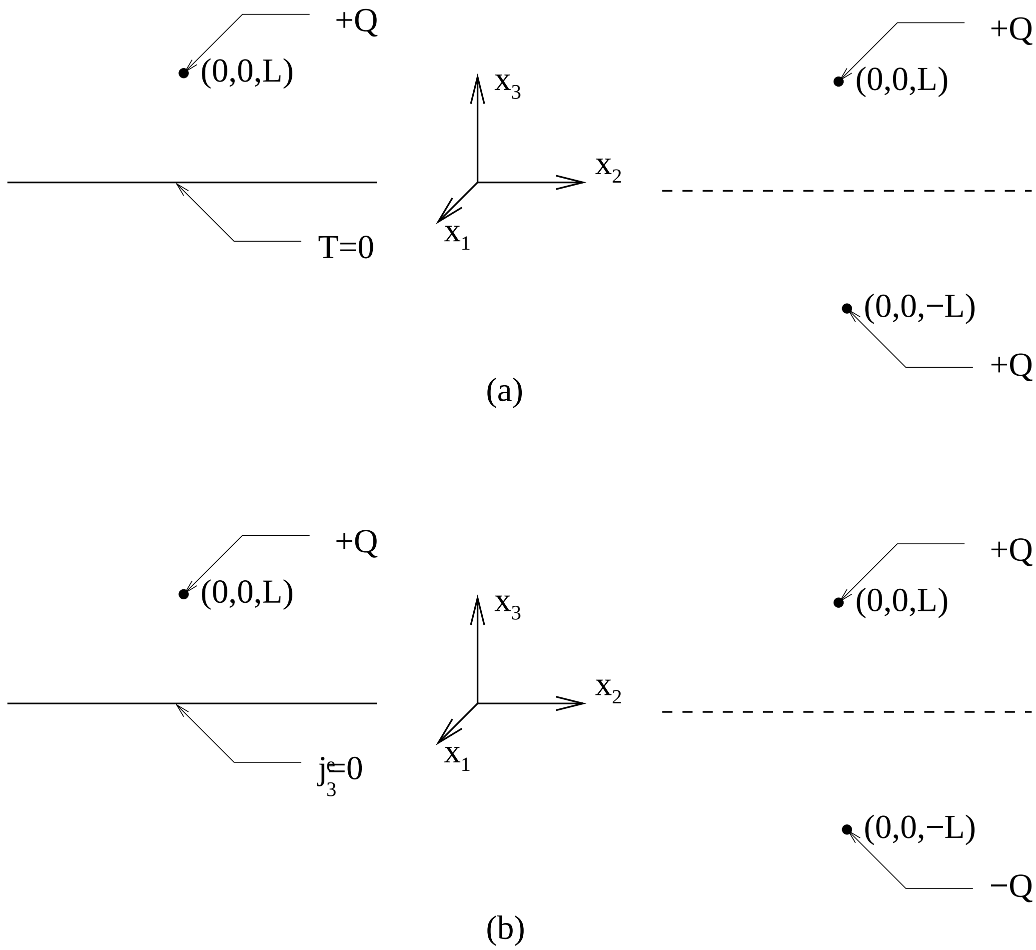


Figure 5.6: Greens functions for source near a wall with (a) zero temperature conditions at the wall and (b) zero flux conditions at the wall.

condition is called the Neumann Greens function G_N,

$$G_N(\mathbf{x} = \frac{1}{4\pi K|\mathbf{x} - \mathbf{x}_s|} + \frac{1}{4\pi K|\mathbf{x} - \mathbf{x}_I|} \tag{5.56}$$

A similar procedure could be used for more complicated geometries. The Greens function for a source in a corner with zero flux conditions, as shown in figure 5.7(a), could be obtained by using four sources of equal strength placed symmetrically in an infinite domain. Similarly, the Greens function for a source in a corner with zero temperature conditions, as shown in figure 5.7(b), could be obtained by placing two sources and two sinks symmetrically in an infinite domain. The Greens function for a source in a finite channel, as shown in figure 5.8, would require an infinite number of sources.

5.2.5 Greens function for a sphere

The Dirichlet Greens function G_D for a sphere is the solution for the temperature field due to a source of unit strength which satisfies the zero temperature condition at the surface of the sphere. This Greens function can be derived using spherical coordinates (r, θ, ϕ). Consider a source of strength 1, which is located, without loss of generality, at $\mathbf{x}_S = (0, 0, r)$ in a sphere of radius 1, as shown in figure 5.9. The image, by symmetry has to be located along the line joining the source point and the origin, at $\mathbf{x}_I = (0, 0, r')$, but the strength Q_I can, in general, be different from that of the source. The temperature at a point on the surface, $\mathbf{x} = (r, \theta, \phi)$ in spherical coordinates, or $\mathbf{x} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$, due to the source and sink, is given by

$$\begin{aligned}
T &= \frac{1}{4\pi K|\mathbf{x} - \mathbf{x}_I|} + \frac{Q}{4\pi K|\mathbf{x} - \mathbf{x}_I|} \\
&= \frac{1}{4\pi K}\left(\frac{1}{(\sin(\theta)^2\cos(\phi)^2 + \sin(\theta)^2\sin(\phi)^2 + (\cos(\theta) - r)^2)^{1/2}}\right. \\
&\quad \left. + \frac{Q}{(\sin(\theta)^2\cos(\phi)^2 + \sin(\theta)^2\sin(\phi)^2 + (\cos(\theta) - r')^2)^{1/2}}\right) \\
&= \frac{1}{4\pi K}\left(\frac{1}{(1 + r^2 - 2r\cos(\theta))^{1/2}} + \frac{Q}{(1 + r'^2 + 2r'\cos(\theta))^{1/2}}\right)
\end{aligned} \tag{5.57}$$

The values of Q and r' are determined from the boundary condition on the surface of the sphere, which requires that $T = 0$ for all values of θ and ϕ.

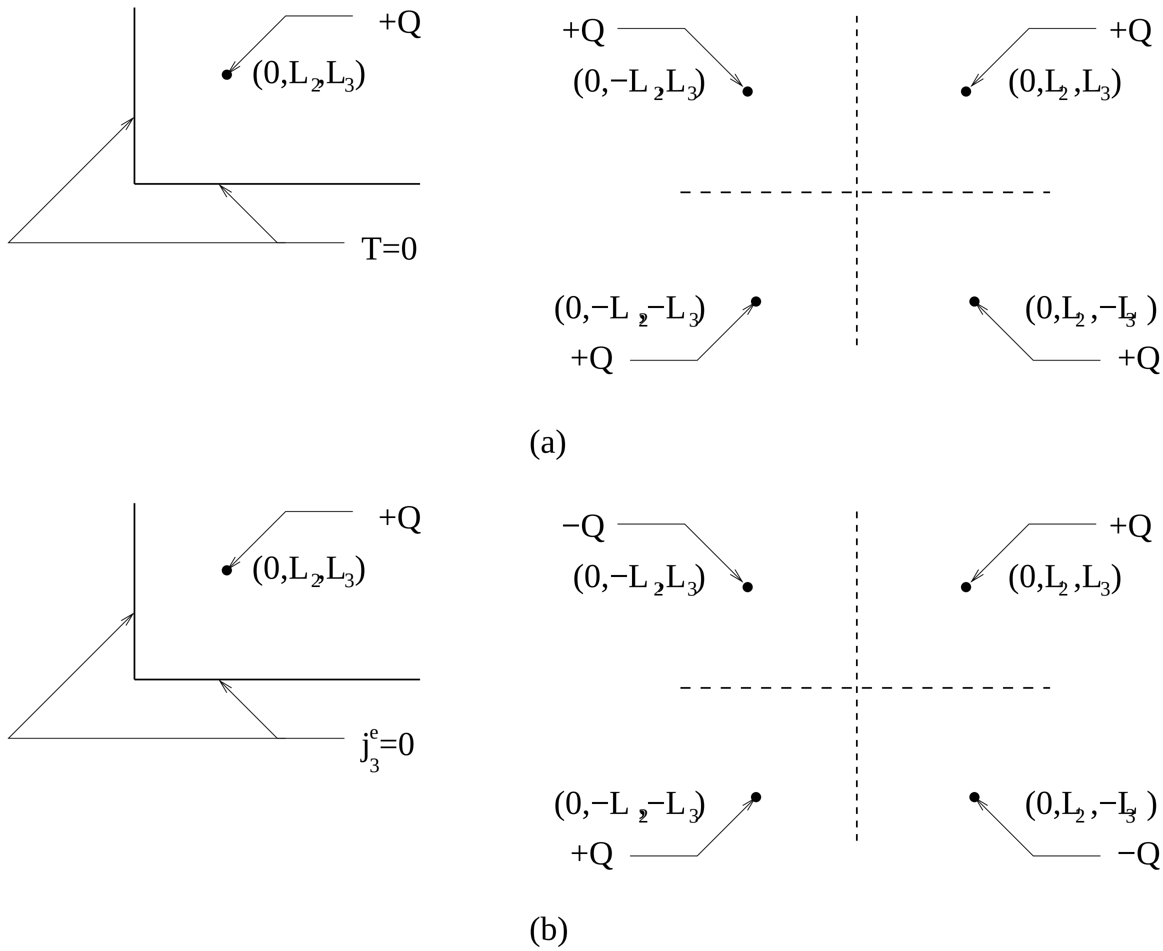


Figure 5.7: Greens functions for source near a corner with (a) zero temperature conditions at the walls and (b) zero flux conditions at the walls.

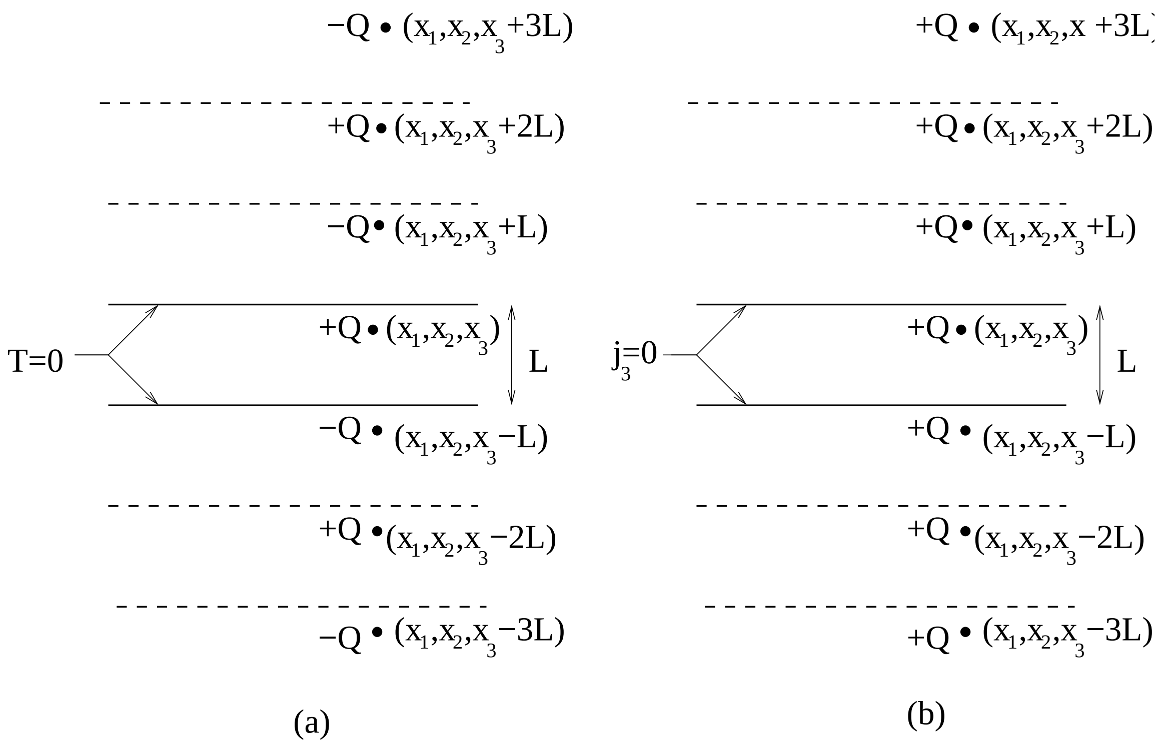


Figure 5.8: Greens functions for source in a channel of finite width with (a) zero temperature conditions at the walls and (b) zero flux conditions at the walls.

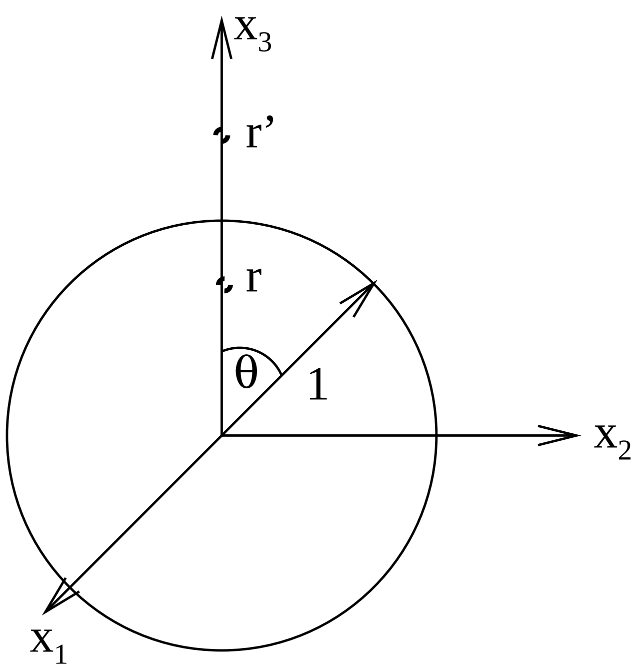


Figure 5.9: Greens function for a sphere.

This condition is satisfied only if

$$\begin{aligned} Q^2(1+r^2) &= 1+r'^2 \\ 2Q^2 r &= 2r' \end{aligned} \tag{5.58}$$

From these two conditions, the solution for r' and Q are

$$\begin{aligned} r' &= r^{-1} \\ Q &= r^{-1} \end{aligned} \tag{5.59}$$

Thus, the solution for the Greens function is

$$T = \frac{1}{4\pi K|\mathbf{x}-\mathbf{x}_I|} + \frac{1}{4\pi K r|\mathbf{x}-\mathbf{x}_I|} \tag{5.60}$$

where $\mathbf{x}_I = (0,0,1/r)$.

5.2.6 Temperature distribution due to a dipole

Consider a source and sink of energy, of equal strength, $\pm Q$ of energy per unit time, located at the positions $(0,0,L)$ and $(0,0,-L)$, as shown in figure 5.10. The temperature field due to the combination of source and sink is given by

$$T(\mathbf{x}) = \frac{Q}{4\pi K(x^2+y^2+(z-L)^2)^{1/2}} + \frac{-Q}{4\pi K(x^2+y^2+(z+L)^2)^{1/2}} \tag{5.61}$$

If the distance of the observation point from the origin is sufficiently large, so that the $(x,y,z) \gg L$, the temperature field is given by

$$\begin{aligned} T(\mathbf{x}) &= \frac{Q}{4\pi K}\frac{2Lz}{(x^2+y^2+z^2)^{1/2}} \\ &= \frac{2QL}{4\pi K}\frac{\cos(\theta)}{r^2} \\ &= \frac{2QL}{4\pi K} r^{-2} P_1(\cos(\theta)) \end{aligned} \tag{5.62}$$

Thus, the temperature distribution due to the combination of a source and a sink, in the limit where the distance between the two reduces to zero, is identical to the second spherical harmonic solution that was obtained for the Laplace equation. Similarly, it can be shown that the combination of two sources and two sinks, arranged in such a way that the net dipole moment is zero, corresponds to the third spherical harmonic solution for the Laplace equation.

5.2.7 Boundary integral technique

In order to solve the steady temperature equation,

$$\nabla^2 T = -S(\mathbf{x}) \tag{5.63}$$

we can use a Greens function

$$\nabla^2 G = \delta(\mathbf{x}) \tag{5.64}$$

which is defined according to the required boundary conditions on the domain, as follows.

$$\begin{aligned} &\int dV' \nabla.(T(\mathbf{x}')\nabla G(\mathbf{x}-\mathbf{x}') - G(\mathbf{x}-\mathbf{x}')\nabla' T(\mathbf{x}')) \\ &= \int dV' (T(\mathbf{x}')\nabla'^2 G(\mathbf{x}-\mathbf{x}') - G(\mathbf{x}-\mathbf{x}')\nabla'^2 T(\mathbf{x}')) \\ &= \end{aligned} \tag{5.65}$$

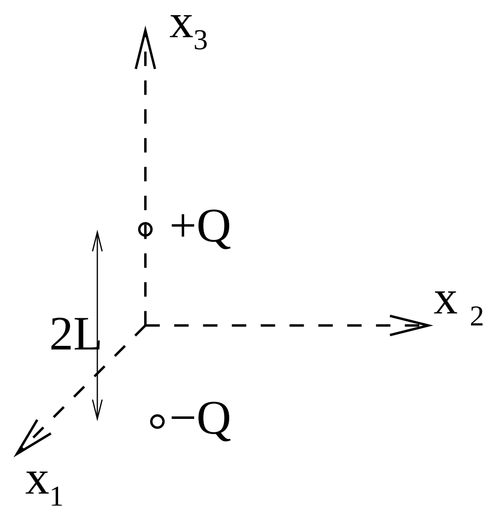


Figure 5.10: Temperature field due to a source and a sink.

The left side of the above equation can be written as an integral over the boundary of the domain using the divergence theorem,

$$\begin{aligned} & \int dV' \nabla . (T(\mathbf{x}') \nabla G(\mathbf{x} - \mathbf{x}') - G(\mathbf{x} - \mathbf{x}') \nabla' T(\mathbf{x}')) \\ = & \int dS' \mathbf{n} . (T(\mathbf{x}') \nabla G(\mathbf{x} - \mathbf{x}') - G(\mathbf{x} - \mathbf{x}') \nabla' T(\mathbf{x}')) \end{aligned} \tag{5.66}$$

where $\mathbf{n}$ is the boundary of the domain. Therefore, the equation 5.66 reduces to

$$\begin{aligned} & T(\mathbf{x}) + \int dV' G(\mathbf{x} - \mathbf{x}') S(\mathbf{x}') \\ = & \int dS' \mathbf{n} . (T(\mathbf{x}') \nabla G(\mathbf{x} - \mathbf{x}') - G(\mathbf{x} - \mathbf{x}') \nabla' T(\mathbf{x}')) \end{aligned} \tag{5.67}$$

5.2.8 Greens function for the unsteady diffusion equation

5.3 Problems

1. Derive the harmonic expansion for a two dimensional cylindrical coordinate system with coordinates (r, θ).

   (a) Use separtion of variables to solve the equation $K\nabla^2 T = 0$ in cylindrical co-ordinates.

   (b) For a point source, solve the heat equation $K\nabla^2 T = Q\delta(\mathbf{x})$ in cylindrical coordinates, to obtain the temperature distribution due to a point source.

   (c) What is the temperature field when two sources are located as shown in figure 1(a) and 1(b), and $L \ll r$? Compare with the second terms in the cylindrical harmonic expansion.

   (d) What is the temperature field when four sources are located as shown in figures 1(c), and (d)? Compare with the third terms in the cylindrical harmonic expansion.

   (e) Determine the second and third terms in the harmonic expansion by successively taking gradients of the temperature field due to the point source.

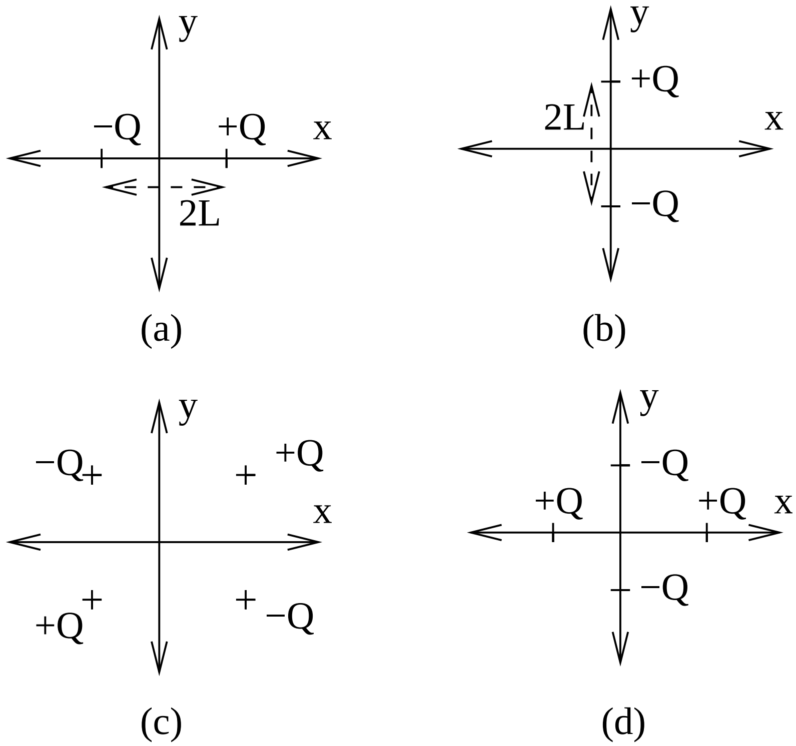
2. Determine the effective thermal conductivity for a dilute array of infinitely long circular cylinders along the plane perpendicular to the axis of the cylinders, when the area fraction of the cylinders is ϕ. Use the following steps.

   (a) Consider an infinitely long cylinder with conductivity K_p in a matrix of conductivity K_m, and determine the temperature field around the cylinder when a uniform gradient T' is imposed in the x direction perpendicular to the axis of the cylinder.

   (b) Write the heat flux as the sum of the flux over the matrix and the sum over the cylinders. When the array is dilute, write the integral as the sum over one cylinder, and determine the thermal conductivity.

   (c) What is the effective conductivity along the axis of the cylinders?

3. A point source of heat of strength Q (in units of heat energy per unit time) is placed at a distance L from a wall.



(a) If the wall is perfectly conducting, so that the flux lines at the wall are perpendicular to the wall as shown in Figure 1(a), determine the temperature as a function of position.

(b) If the wall is perfectly insulating, so that the flux lines at the wall are parallel to the wall as shown in Figure 1(b), determine the temperature as a function of position.

(c) If the wall is not perfectly conducting, but only a fraction f of the heat on the wall penetrates it, while a fraction $(1 - f)$ does not penetrate the wall, determine the temperature field as a function of position.

4. A heater coil in the form of a ring of radius a in the $x-y$ plane generates heat Q per unit length of the coil per unit time, as shown in figure 2.

   (a) If the heater is placed in an unbounded medium of thermal conductivity K, write an equation for the temperature as a function of position in the medium.

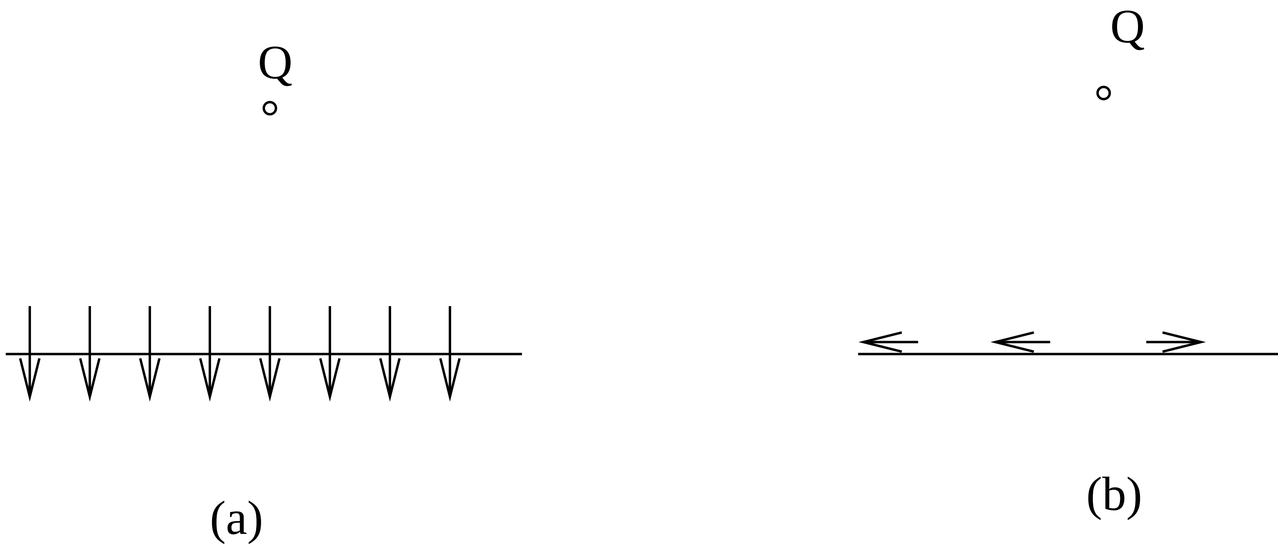


Figure 5.11:

(b) Plot the temperature as a function of position along the symmetry axis of the heater (z axis in the figure). Simplify the expressions for the temperature for $z \ll a$ and $z \gg a$. What does the expression for $z \gg a$ correspond to?

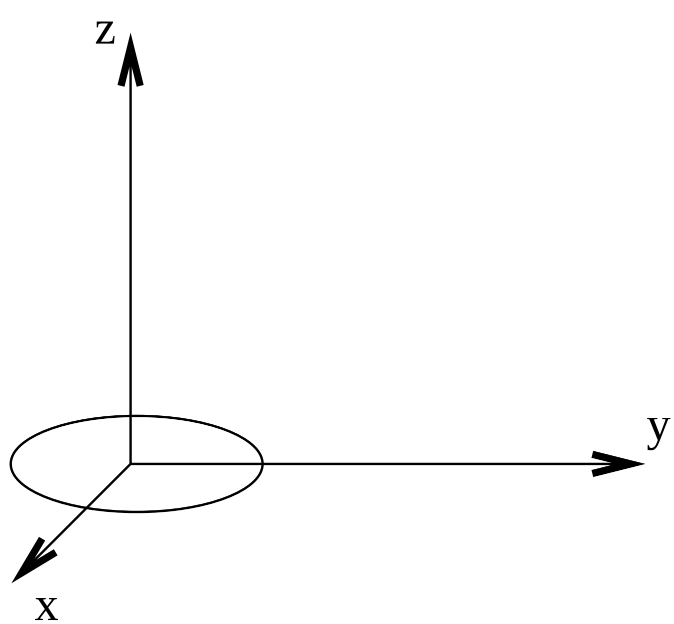


Figure 5.12: