Chapter 1

Forced convection at high Peclet number:

1.1 Streaming past a spherical object:

The system consists of a sphere whose surface is maintained at a temperature T_0 in a fluids in which the temperature is T_∞ in the limit $r \to \infty$. There is a steady flow of fluids past the sphere, and the fluid velocity in the z direction is U_∞ at a large distance from the sphere, as shown in figure 1.1. As usual, it is convenient to work in terms of a scaled temperature $\Theta = (T - T_\infty)/(T_0 - T_\infty)$. The convection-diffusion equation in terms of the scaled temperature θ is given by,

$$\mathbf{u}.\nabla\Theta = \alpha\nabla^2\Theta \tag{1.1}$$

The boundary conditions, expressed in terms of the scaled temperature Θ,

$$\begin{aligned} \Theta &= 1 \text{ at } r^* = 1 \\ \Theta &\to 0 \text{ for} r^* \to \infty \end{aligned} \tag{1.2}$$

It is convenient to scale the lengths in the problem by the sphere radius R, and the velocity by the free stream velocity U_∞. In terms of the scaled coordinates, the convection-diffusion equation can be written as,

$$\mathrm{Pe}\mathbf{u}^*.\nabla^*\theta = \nabla^{*2}\theta \tag{1.3}$$

where $\mathbf{u}^* = (\mathbf{u}/U_\infty)$, $r^* = (r/R)$, ∇^* is the gradient operator defined in terms of the scaled radius r^* and the angular coordinates θ and ϕ, and

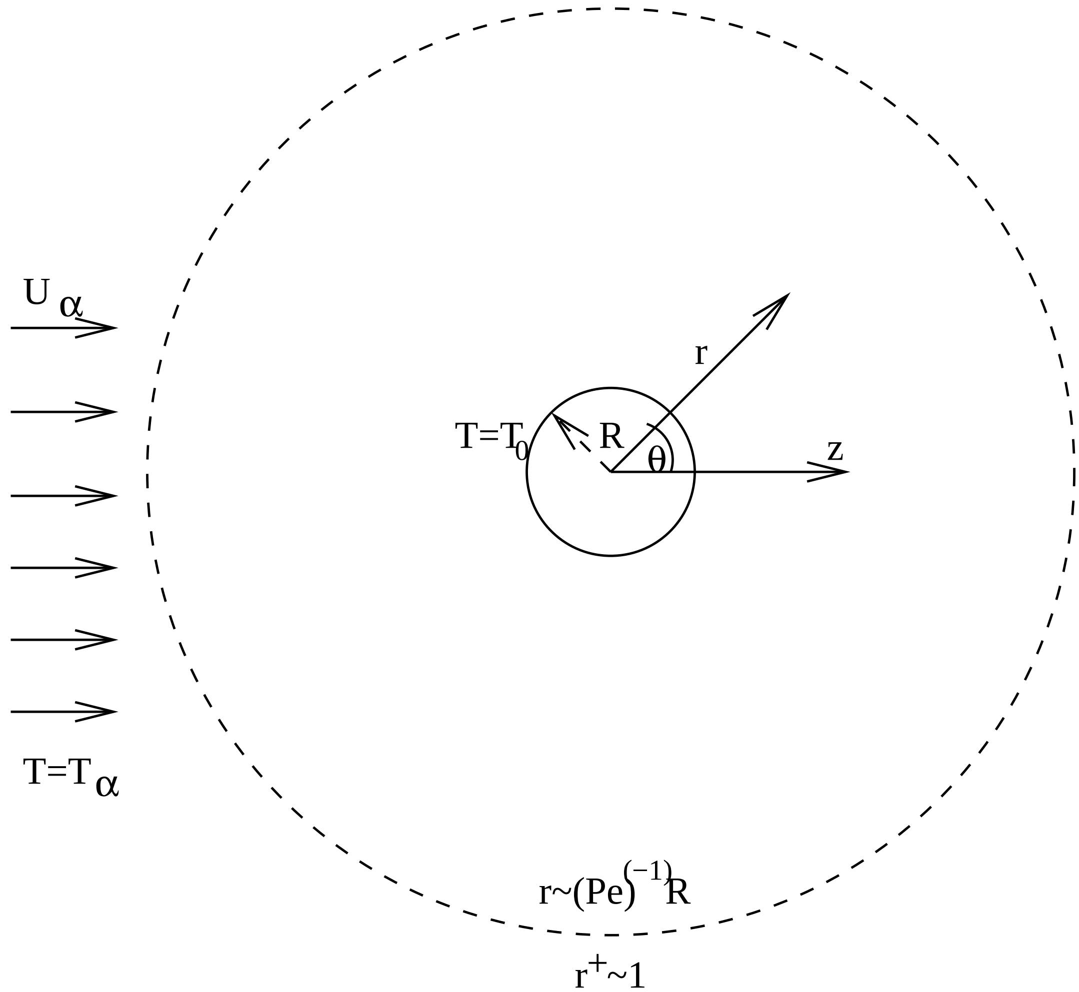

Figure 1.1: Forced convection due to the streaming flow past a sphere of radius R. The velocity at a large distance from the sphere is U_∞ in the z direction, the temperature at the surface of the sphere is maintained at T_0, while the temperature of the fluid at a large distance from the sphere is T_∞.

$\mathrm{Pe} = (U_\infty R/\alpha)$ is the Peclet number. The velocity field for the flow around a sphere at low Reynolds number, in a spherical coordinate system, is given by a solution of the Stokes equations, which will be solved a little later. The components of the scaled velocity $\mathbf{u}^*$, expressed in terms of the scaled radius r^*, are,

$$u_r^* = \left[1 - \frac{3}{2}\left(\frac{1}{r^*}\right) + \frac{1}{2}\left(\frac{1}{r^*}\right)^3\right]\cos(\theta) \tag{1.4}$$

$$u_\theta = -\left[1 - \frac{3}{4}\left(\frac{1}{r^*}\right) - \frac{1}{4}\left(\frac{1}{r^*}\right)^3\right]\sin(\theta) \tag{1.5}$$

where r^* is the distance from the center of the sphere, scaled by the radius of the sphere, and θ is the angle made by the radius vector with the z coordinate. Note that the configuration and boundary conditions are symmetric in the meridional ϕ coordinate, and so the temperature field is independent of the ϕ coordinate.

We consider the limit of small Peclet number, and therefore it is convenient to expand the temperature θ in an asymptotic series in the Peclet number.

$$\Theta = \Theta_0 + \mathrm{Pe}\Theta_1 + \mathrm{Pe}^2\Theta_2 + \ldots \tag{1.6}$$

where Θ_0 is the temperature distribution in the absence of convective effects, Θ_1 is the first correction to the temperature distribution due to convection, etc. This expansion is inserted into the conservation equation 1.3, to obtain the leading order and higher corrections to the temperature distribution. The equations for the leading order and first correction to the temperature distribution are,

$$\nabla^{*2}\Theta_0 = 0 \tag{1.7}$$

$$u_r^*\frac{\partial\Theta_0}{\partial r} + u_\theta^*\frac{\partial\Theta_0}{\partial\theta} = \nabla^{*2}\Theta_1 \tag{1.8}$$

The boundary conditions for the leading order and first correction to the temperature are obtained by inserting the expansion 1.6 into the equations for the boundary conditions 1.2,

$$\Theta_0 + \mathrm{Pe}\Theta_1 + \ldots = 1 \text{ at } r^* = 1 \tag{1.9}$$

$$\Theta_0 + \mathrm{Pe}\Theta_1 + \ldots = 0 \text{ for } r^* \to \infty \tag{1.10}$$

Equating terms of equal powers of Pe on the left and right sides of the above equations, we obtain the boundary conditions for the leading order and first correction to the temperature field,

$$\Theta_0 = 1 \text{ at } r^* = 1 \tag{1.11}$$

$$\Theta_0 \to 0 \text{ at } r^* \to \infty \tag{1.12}$$

$$\Theta_1 = 0 \text{ at } r^* = 1 \tag{1.13}$$

$$\Theta_1 \to 0 \text{ at} r^* \to \infty \tag{1.14}$$

The solution for the leading order equation can be obtained quite easily, since the configuration is spherically symmetric in the absence of flow, and the boundary conditions are also spherically symmetric. The leading order diffusion equation reduces to,

$$\frac{1}{r^{*2}} \frac{d}{dr^*} \left(r^{*2} \frac{d\Theta_0}{dr^*} \right) = 0 \tag{1.15}$$

This equation can be easily solved with the boundary condition $\Theta_0 = 1$ at $r^* = 1$, to obtain the leading approximation to the temperature field,

$$\Theta_0 = \frac{1}{r^*} \tag{1.16}$$

The flux at the surface can be easily obtained in terms of this leading correction to the temperature,

$$\begin{aligned} \mathbf{j}_0 &= K\nabla T \\ &= -\frac{K(T_0 - T_\infty)}{R} \nabla^* \Theta_0 \\ &= \frac{K(T_0 - T_\infty)}{R r^{*2}} \mathbf{e}_r \end{aligned} \tag{1.17}$$

The total heat transfer from the sphere is obtained by multiplying the heat flux in the radial direction at $r^* = 1$ by the surface area of the sphere, to obtain

$$Q_0 = 4\pi K (T_0 - T_\infty) R \tag{1.18}$$

The dimensionless total heat transfer rate, which is the Nusselt number, is defined as,

$$\mathrm{Nu} = \frac{2Q}{(4\pi R^2) K (T_0 - T_\infty)/R} \tag{1.19}$$

Therefore, the leading order contribution to the Nusselt number is $\mathrm{Nu}_0 = 2$.

The leading order solution can be inserted into the equation for the first correction, 1.8, to obtain,

$$\begin{aligned} \frac{1}{r^{*2}} \frac{d}{dr^*} \left(r^{*2} \frac{d\Theta_1}{dr^*} \right) + \frac{1}{r^{*2} \sin(\theta)} \frac{d}{d\theta} \left(\sin(\theta) \frac{d\Theta_1}{d\theta} \right) \\ = -\frac{1}{r^{*2}} \left(1 - \frac{3}{2r^*} + \frac{1}{2r^{*3}} \right) \cos(\theta) \end{aligned} \tag{1.20}$$

The above equation is an inhomogeneous linear differential equation, and so the solution can be written as the sum of a particular and general solution. First, we obtain the particular solution. Since the inhomogeneous term on the right side is proportional to $\cos(\theta) = P_1^0(\cos(\theta)$, we would expect the particular solution for Θ_1 to also be proportional to $\cos(\theta)$. Using the form $\Theta_{1p} = F(r^*) \cos(\theta)$, and inserting this form into equation 1.20, we can solve for $F(r^*)$ to obtain the explicit form of the particular solution,

$$\Theta_{1p} = \left(\frac{1}{2} - \frac{3}{4r^*} - \frac{1}{8r^{*3}} \right) P_1(\cos(\theta)) \tag{1.21}$$

In addition, there is a general solution for the homogeneous equation, which is obtained by setting the right side of equation 1.20 equal to zero. Since the linear operator acting on Θ_1 is the Laplacian, the general solution is just the sum of spherical harmonics. Since the leading order solution for the temperature field is proportional to $(1/r^*)$, and the inhomogeneous term on the right side of equation 1.20 is proportional to $P_1^0(\cos(\theta)$ it is sufficient to include just the first two terms in the spherical harmonic expansion in the general solution,

$$\Theta_{1g} = \left(A_0 + \frac{B_0}{r^*} \right) + \left(A_1 r^* + \frac{B_1}{r^{*2}} \right) P_1^0(\cos(\theta) \tag{1.22}$$

With this, the final solution for the temperature field becomes,

$$\Theta_1 = \left(A_0 + \frac{B_0}{r^*} \right) + \left(A_1 r^* + \frac{B_1}{r^{*2}} \right) P_1^0(\cos(\theta) + \left(\frac{1}{2} - \frac{3}{4r^*} - \frac{1}{8r^{*3}} \right) P_1(\cos(\theta)) \tag{1.23}$$

Imposing the boundary condition $\Theta_1 = 0$ at $r^* = 0$ (1.13), we obtain,

$$\begin{aligned} A_0 + B_0 &= 0 \\ A_1 + B_1 &= (3/8) \end{aligned} \tag{1.24}$$

The boundary condition $\Theta_1 = 0$ at $r^* \to \infty$ (1.14) requires that

$$A_1 = 0 \tag{1.25}$$

since A_1 multiplies a term that increases proportional to r^* in the limit $r^* \to \infty$. It would also appear that the boundary condition 1.14 also requires that $A_0 = 0$, but that turns out to not be the case, for reasons that will become clear shortly. Therefore, the final expression for the temperature field correct to $O(\mathrm{Pe})$, $\Theta = \Theta_0 + \mathrm{Pe}\Theta_1$, is,

$$\Theta = \frac{1}{r^*} + \mathrm{Pe}\left[B_0\left(\frac{1}{r^*} - 1\right) + \left(\frac{1}{2} - \frac{3}{4r^*} + \frac{3}{8r^{*2}} - \frac{1}{8r^{*3}}\right)P_1(\cos(\theta))\right] \tag{1.26}$$

It should by now have become apparent that when we try to apply the boundary conditions 1.14 at $r^* \to \infty$, we find that there is a constant term on the right side, $(1/2)P_1(\cos(\theta))$, which cannot be matched with any of the other terms. Therefore, it is not possible to satisfy all the boundary conditions simultaneously. This signifies that there is something wrong with the original approach to the outer problem in the limit $r^* \to \infty$.

The origin of the problem can be explained as follows. We once again examine the left and right sides of the conservation equation 1.3 in the limit $r^* \to \infty$. On the left side the velocity u_r^* tends to a constant in the limit $r^* \to \infty$, while the gradient of the leading order solution for the temperature scales as $(1/r^{*2})$. On the right side, the Laplacian of the leading order temperature scales as $(1/r^{*3})$. Therefore, the ratio of the convective and diffusive terms scales as $(\mathrm{Pe}r^*)$. Even when the Peclet number is small, if we go to sufficiently large distances such that $r^* \sim (1/\mathrm{Pe})$, it is clear that the convective term becomes larger than the diffusive term, and can no longer be neglected while calculating the outer solution. While the leading solution 1.16 is valid for $r^* \ll (1/\mathrm{Pe})$, it is not valid for $r^* \sim (1/\mathrm{Pe})$, and it is necessary to incorporate the convective term in this region to obtain a valid leading order solution. This is done by rescaling the radial coordinate as $r^\dagger = \mathrm{Pe}r^*$, so that the convective and diffusive terms are of equal magnitude in this region. We first re-express the velocity components in terms of the coordinate $r^\dagger$,

$$\begin{aligned} u_r^* &= \left[1 - \frac{3}{2}\left(\frac{\mathrm{Pe}}{r^\dagger}\right) + \frac{1}{2}\left(\frac{\mathrm{Pe}}{r^\dagger}\right)^3\right]\cos(\theta) \\ &\simeq \cos(\theta) \end{aligned} \tag{1.27}$$

$$\begin{aligned} u_\theta &= -\left[1 - \frac{3}{4}\left(\frac{\mathrm{Pe}}{r^\dagger}\right) - \frac{1}{4}\left(\frac{\mathrm{Pe}}{r^\dagger}\right)^3\right]\sin(\theta) \\ &\simeq -\sin(\theta) \end{aligned} \tag{1.28}$$

In the expressions for the velocity, we have retained only the leading order terms in an expansion in the small parameter Pe. This is inserted into the conservation equation 1.3, to obtain,

$$\cos(\theta)\frac{\partial\Theta_0^\dagger}{\partial r^\dagger} - \sin(\theta)\frac{1}{r^\dagger}\frac{\partial\Theta_0^\dagger}{\partial\theta} = \nabla^{\dagger 2}\Theta_0^\dagger \tag{1.29}$$

where $\nabla^\dagger = \mathrm{Pe}\nabla^*$, and we have denoted the solution $\Theta_0^\dagger$ with a superscript † in order to indicate that this is the leading order solution for $r^\dagger \sim 1$.

To solve the above equation 1.30, it is convenient to introduce a transformation

$$\Theta_0^\dagger = \exp(r^\dagger\cos(\theta)/2)\Phi(r^\dagger,\theta) \tag{1.30}$$

Substituting equation 1.31 in equation 1.30, we obtain,

$$\nabla^{\dagger 2}\Phi - \frac{1}{4}\Phi = 0 \tag{1.31}$$

The solution of the above equation can be simplified by anticipating the matching condition to be satisfied in the limit $r^\dagger \ll 1$ and $r^* \gg 1$. In this region, the inner solution Θ_0 is proportional to $(1/r^*)$ and is spherically symmetric. The outer solution, $\Theta_0^\dagger = \Phi\exp(r^\dagger\cos(\theta)/2)$, is just proportional to Φ for $r^\dagger \ll 1$. Therefore, if the outer solution is to match with the spherically symmetric inner solution, we require that Φ should also be spherically symmetric. With this, the equation for Φ reduces to,

$$\frac{1}{r^{\dagger 2}}\frac{d}{dr^\dagger}\left(r^{\dagger 2}\frac{d\Phi}{dr^\dagger}\right) - \frac{1}{4}\Phi = 0 \tag{1.32}$$

The solution for this, which satisfies the boundary condition that the temperature goes to zero as $r^\dagger \to \infty$, is

$$\Phi = A^\dagger\frac{\exp(-r^\dagger/2)}{r^\dagger} \tag{1.33}$$

Therefore, the final solution for the temperature field in the outer region is of the form,

$$\Theta_0^\dagger = A^\dagger\frac{\exp(-(r^\dagger/2)(1-\cos(\theta)))}{r^\dagger} \tag{1.34}$$

The first correction to the temperature field, $\Theta_1^\dagger$, the region $r^\dagger \sim 1$, can be determined in a manner similar to that carried out previously to obtain equation 1.20. However, the explicit form of the equation is not necessary in the present analysis, and we only note that $\Theta_1^\dagger$ has to be small compared to $\Theta_0^\dagger$ in the small Peclet number limit.

The constant A has to be determined by 'asymptotic matching' of the two solutions 1.26 and 1.35 which were obtained in different domains. The solution $\Theta_0 = (1/r^*)$ (equation 1.26) for $r^* \sim 1$, which is equivalent to $r^\dagger = \text{Pe}^{-1} r^* \ll 1$. The solution $\Theta_0^\dagger = (A/r^\dagger) \exp\left(-(r^\dagger/2)(1-\cos(\theta))\right)$ (equation 1.35) was obtained for $r^\dagger \sim 1$, or $r^* = \text{Pe} r^\dagger \gg 1$. Since these are solutions for the same temperature field obtained in different domains, consistency requires that these exhibit the same behaviour in the intermediate regime where, simultaneously, $r^\dagger \ll 1$ and $r^* \gg 1$. Formally, this is carried out by setting $r \sim R\text{Pe}^{-\alpha}$, where α is in the interval $0 < \alpha < 1$, so that $r^* = (r/R) = \text{Pe}^{-\alpha} \gg 1$, and $r^\dagger = (r/R)\text{Pe} = \text{Pe}^{1-\alpha} \ll 1$. We first examine the solution equation 1.26 for Θ. The leading terms in the equation are the first term on the right, which is proportional to $\text{Pe}^{-\alpha}$, and the contributions to the second term which are independent of r^*, which are proportional to Pe. Therefore, the leading approximation for Θ in the limit $r^* \to \infty$ is,

$$\Theta = \frac{R}{r} - \text{Pe} B_0 + \frac{\text{Pe}\cos(\theta)}{2} \tag{1.35}$$

Next, we examine equation 1.35 for $\Theta_0^\dagger$ in the limit $r^\dagger \ll 1$. The exponential in equation 1.35 is expanded in a Taylor series to linear order in $r^\dagger$, and the substitution $r^\dagger = \text{Pr}(r/R)$ is used, to obtain,

$$\Theta_0^\dagger = \frac{A^\dagger}{\text{Pe}} Rr + A^\dagger \frac{(\cos(\theta) - 1)}{2} \tag{1.36}$$

Clearly, the largest contribution to $\Theta_0^\dagger$ is due to the first term on the right side of equation 1.37. Equating this with the first term on the right side of equation 1.36 for Θ, we find that $A^\dagger = \text{Pe}$. Therefore, the equation for $\Theta_0^\dagger$ becomes,

$$\begin{aligned} \Theta_0^\dagger &= \text{Pe}\frac{\exp\left(-(r^\dagger/2)(1-\cos(\theta))\right)}{r^\dagger} \\ &\simeq \frac{R}{r} + \text{Pe}\frac{(\cos(\theta)-1)}{2} \ \text{for} r^\dagger \ll 1 \end{aligned} \tag{1.37}$$

The $O(\mathrm{Pe})$ term in equation 1.37 is now amtched with the $O(\mathrm{Pe})$ term in equation 1.35 to determine the value of the constant B_0. It can be easily seen that the $O(\mathrm{Pe})$ contributions to equations 1.35 and 1.37 are identical in form for $B_0 = 1/2$, and therefore the solution for Θ_0 which matches with the outer solution $\Theta_0^{\dagger}$ is obtained by setting $B_0 = 1/2$ in equation 1.26,

$$\Theta = \frac{1}{r^*} + \mathrm{Pe}\left[\frac{1}{2}\left(\frac{1}{r^*} - 1\right) + P_1(\cos(\theta))\left(\frac{1}{2} - \frac{3}{4r^*} + \frac{3}{8r^{*2}} - \frac{1}{8r^{*3}}\right)\right] \quad (1.38)$$

The total heat flux from the sphere can now be calculated,

$$\begin{aligned} Q &= \int_S dS \left(-K \left.\frac{dT}{dr}\right|_{r=R}\right) \\ &= -\frac{K(T_0 - T_\infty)}{R} \int_S dS \left.\left(\frac{d\Theta}{dr^*}\right)\right|_{r^*=1} \end{aligned} \quad (1.39)$$

The integral over the spherical surface of the terms proportional to $\cos(\theta)$ on the right side of equation 1.38 are identically zero, and so the expression for the heat flux reduces to,

$$\begin{aligned} Q &= -\frac{K(T_0 - T_\infty)}{R} \int_S dS \left.\left(-\frac{1}{r^{*2}} - \frac{\mathrm{Pe}}{2r^{*2}}\right)\right|_{r^*=1} \\ &= 4\pi K(T_0 - T_\infty) R\,(1 + (\mathrm{Pe}/2)) \end{aligned} \quad (1.40)$$

The Nesselt number can now be evaluated,

$$\begin{aligned} \mathrm{Nu} &= \frac{2Q}{(4\pi R^2)(K(T_0 - T_\infty)/R)} \\ &= 2(1 + \mathrm{Pe}/2) \end{aligned} \quad (1.41)$$

The higher order terms in the expansion can also be calculated using matched asymptotic expansions. The expression for the Nusselt number that incorporates higher order corrections is,

$$\mathrm{Nu} = 2\left(1 + \frac{1}{2}\mathrm{Pe} + \frac{1}{2}\mathrm{Pe}^2\log(\mathrm{Pe}) + 0.41465\mathrm{Pe}^2 + \frac{1}{4}\mathrm{Pe}^3\log(\mathrm{Pe}) + O(\mathrm{Pe}^3)\right) \quad (1.42)$$

1.2 Streaming past an object of arbitrary shape:

In the case of streaming past an object of arbitrary shape, we would expect the expansion for the Nusselt number, in the limit of small Peclet number, to be of the form,

$$\mathrm{Nu} = \mathrm{Nu}_0 + \mathrm{Pe}\mathrm{Nu}_1 + \ldots \tag{1.43}$$

While it is necessary to determine the leading order contribution to the dimensionless heat flux, Nu_0, by solving the heat conduction equation for the specific shape under consideration, it can be shown that the first correction Nu_1 is related to the leading order dimensionless heat flux by

$$\mathrm{Nu}_1 = \frac{1}{4}\mathrm{Nu}_0^2 \tag{1.44}$$

This is an example of diffusion dominated processes where results can be obtained just from a knowledge of the 'far field' temperature distribution which is only a function of the most slowly decaying component of the temperature field, without a knowledge of the temperature distribution in the vicinity of the object. We first examine some general features of the 'far field' solution, and show how these can be used to obtain the first correction to the Nusselt number.

A spherical coordinate system is used, as shown in figure 1.2, and the scaled radial coordinate is defined as $r^* = (r/R)$, where R is now some characteristic length of the object, which is assumed to be the radius of the sphere circumscribing the object. It is easy to see that there are some features of the solutions for streaming past a sphere which are easy to extend to objects of arbitrary shape, in which the boundary condition for the scaled temperature is $\Theta = 1$ on the surface of the object, and $\Theta = 0$ at a large distance from the object. The conservation equations for the leading order solution Θ_0 and the first correction Θ_1 are still given by equations 1.7 and 1.8. The boundary conditions in the limit $r^* \to \infty$, 1.12 and 1.14, remain unchnaged, whereas the boundary conditions 1.11 and 1.13 are now applied on the surface of the object,

$$\Theta_0 = 1 \text{ on } S \tag{1.45}$$

$$\Theta_1 = 0 \text{ on } S \tag{1.46}$$

where S is the equation of the surface of the object.

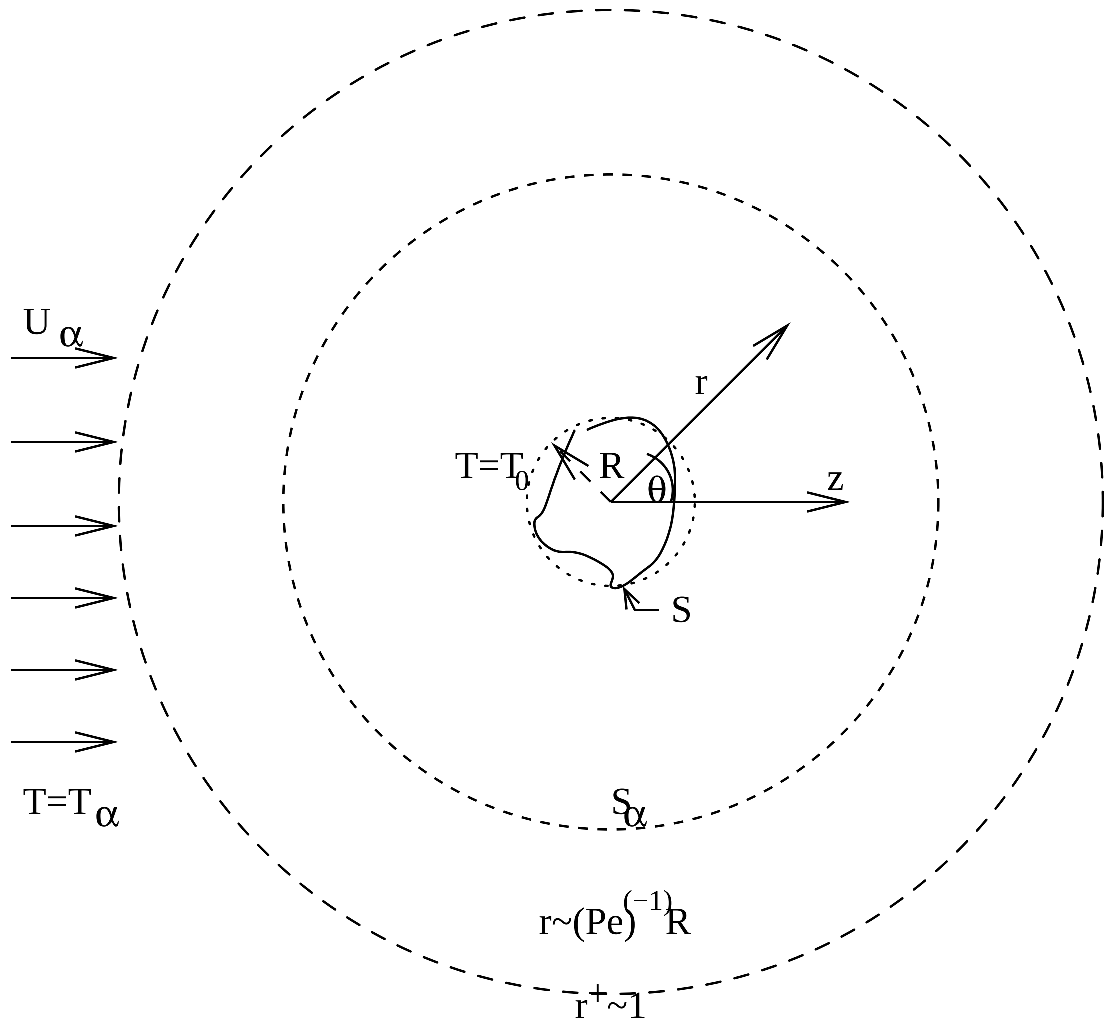

Figure 1.2: Forced convection due to the streaming flow past an object of arbitrary shape. R is the radius of the sphere circumscribing the object, the velocity at a large distance from the sphere is U_∞ in the z direction, the temperature at the surface of the sphere is maintained at T_0, while the temperature of the fluid at a large distance from the sphere is T_∞. The 'far-field' calculations are done over a surface S_∞ which is at a radius $r \gg R$ but $r \ll \mathrm{Pe}^{-1} R$ as shown.

In the absence of spherical symmetry, the solution for Θ_0 obtained by solving equation 1.7 is a linear combination of the spherical harmonics,

$$\Theta_0 = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{nm} r^{-(n+1)} Y_n^m(\theta, \phi) \tag{1.47}$$

where the coefficients A_{nm} are suitably chosen to satisfy the boundary conditions on the surface of the object S. However, rate at which energy is emitted from the object depends only on the first term in the series. If we consider a sphere circumscribing the object, the outward rate of transfer of energy from this sphere is equal to that from the object, since there is no source of energy in the fluid. Therefore, at any distance r from the center of the object, the total energy conducted per unit time can therefore be calculated as,

$$\begin{aligned} Q &= \int_S dS \left(-K \frac{dT}{dr} \right) \\ &= -\frac{K(T_0 - T_\infty)}{R} \int_S dS \left(\frac{\partial \Theta}{\partial r} \right) \end{aligned} \tag{1.48}$$

The leading contribution to the heat transmitted by the object, Q_0, due to the leading order solution for the temperature in the high Peclet number limit, Θ_0, is

$$\begin{aligned} Q_0 &= \frac{K(T_0 - T_\infty)}{r^2} (4\pi r^2) A_{00} \\ &= 4\pi K (T_0 - T_\infty) A_{00} \end{aligned} \tag{1.49}$$

The heat flux depends only on the first term in the expansion in equation 1.48, because the integral of all other terms over the surface of a sphere is identically zero. The leading order contribution to the Nusselt number can now be expressed in terms of A_{00},

$$\begin{aligned} \mathrm{Nu}_0 &= \frac{2Q_0}{S(K(T_0 - T_\infty)/R)} \\ &= \frac{2SA_{00}}{R} \end{aligned} \tag{1.50}$$

The expression 1.49 for the leading order temperature can be rewritten by expressing A_{00} in terms of Nu_0,

$$\Theta_0 = \frac{\mathrm{Nu}_0 R}{2Sr} + \sum_{n=1}^{\infty} \sum_{m=-n}^{n} A_{nm} r^{-(n+1)} Y_n^m(\theta, \phi) \tag{1.51}$$

The solution for $\Theta_0^\dagger$ is also independent of the shape of the particle, since it depends only on the temperature and velocity fields at a large distance from the particle. The leading order approximation for the fluid velocity in equation 1.29 is just the free stream velocity, $\mathbf{u}^* = \mathbf{e}_r \cos(\theta) + \mathbf{e}_\theta \sin(\theta)$. We did make the assumption, just after equation 1.31 that the function Φ is spherically symmetric. However, this equation is valid for an object of arbitrary shape, because the leading order solution for $\Theta_0^\dagger$ is matched to the term in the solution equation 1.16 for Θ_0 which is proportional to $(1/r^*)$, and this term is spherically symmetric. Therefore, the leading order solution $\Theta_0^\dagger$ equation 1.34 is also valid for objects of arbitrary shape.

Next, we examine the derivation of Θ_1 from equation 1.20. In this equation, if r^* is sufficiently large compared to 1 (but small compared to Pe^{-1}), the fluid velocity can be approximated by the free-stream velocity at a large distance from the object, $\mathbf{u}^* = \mathbf{e}_r \cos(\theta) - \mathbf{e}_\theta \sin(\theta)$. Therefore, the leading approximation for equation 1.20 in the limit $1 \ll (r/R) \ll \mathrm{Pe}^{-1}$ is

$$\begin{aligned} \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Theta_1}{dr}\right) + \frac{1}{r^2\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta_1}{d\theta}\right) \\ = \quad -\frac{U_\infty \mathrm{Nu}_0 S}{2\alpha R r^2}\cos(\theta) \end{aligned} \tag{1.52}$$

This equation can be easily solved to obtain,

$$\Theta_1 = \left(A_0 + \frac{B_0}{r^*}\right) + \frac{U_\infty \mathrm{Nu}_0 S}{4\alpha R} P_1(\cos(\theta)) \tag{1.53}$$

Note that the term (B_0/r^*) is small compared A_0 in the limit $r^* \gg 1$. When we match the above solution with the solution 1.34 in the limit $r^\dagger \ll 1$, we find that $A_0 = (-1/2)$. Therefore, the leading order solution for Θ_1 is,

$$\Theta_1 = \frac{U_\infty \mathrm{Nu}_0 S}{4\alpha R}(P_1(\cos(\theta)) - 1) \tag{1.54}$$

The conservation equations 1.7 and 1.8 are rewritten as,

$$\nabla \mathbf{q}_0 \quad = \quad 0 \tag{1.55}$$

$$\nabla \mathbf{q}_1 \quad = \quad -\mathbf{u}^*.\nabla\Theta_0 \tag{1.56}$$

Note that the above equations 1.54 and 1.55 apply only for $r^\dagger = (Per^*) \ll 1$, since we have shown in the previous section that there is a correction due to convective effects for $r^\dagger \sim 1$. The Nusselt number can be calculated once the equations for the flux are known,

$$\begin{aligned} \mathrm{Nu}_0 &= \frac{2Q_0}{S(K(T_0 - T_\infty)/R)} \\ &= \frac{2}{S(K(T_0 - T_\infty)/R)} \int_S dS \mathbf{q}_0.\mathbf{n} \end{aligned} \tag{1.57}$$

$$\begin{aligned} \mathrm{Nu}_1 &= \frac{2Q_1}{S(K(T_0 - T_\infty)/R)} \\ &= \frac{2}{S(K(T_0 - T_\infty)/R)} \int_S dS \mathbf{q}_1.\mathbf{n} \end{aligned} \tag{1.58}$$

where $\mathbf{n}$ is the unit normal to the surface.

Equation 1.55 for the first correction to the temperature field can be rewritten, noting that $\nabla.\mathbf{u}^* = 0$, to obtain

$$\nabla.(\mathbf{q}_1 + \mathbf{u}\Theta_0) = 0 \tag{1.59}$$

It is convenient to multiply 1.58 by Θ_0, and rewrite the result in the form,

$$\nabla.\left(\Theta_0\mathbf{q}_1 - \Theta_1\mathbf{q}_0 + \frac{1}{2}\Theta_0^2\mathbf{u}\right) = 0 \tag{1.60}$$

Here, we have used the simplifiation

$$\begin{aligned} \Theta_0\nabla.\mathbf{q}_1 &= \nabla.(\Theta_0\mathbf{q}_1) - \mathbf{q}_1.\nabla\Theta_0 \\ &= \nabla.(\Theta_0\mathbf{q}_1) - (\nabla\Theta_1).(\nabla\Theta_0) \\ &= \nabla.(\Theta_0\mathbf{q}_1) - \nabla.(\Theta_1\nabla\Theta_0) + \Theta_1\nabla^2\Theta_0 \\ &= \nabla.(\Theta_0\mathbf{q}_1) - \nabla.(\Theta_1\nabla\Theta_0) \end{aligned} \tag{1.61}$$

The final equality follows from the leading order equation 1.54, since $\nabla\Theta_0$ is zero.

Next, we use the divergence theorem to express equation 1.59 as the difference between two surface integrals, one at the surface of the object S, and the other at a surface S_∞ which is at a large distance from the object $r^* \to \infty$.

$$\int_{S_\infty} dS_\infty \left(\Theta_0 \mathbf{q}_1 - \Theta_1 \mathbf{q}_0 + \frac{1}{2}\Theta_0^2 \mathbf{u}\right).\mathbf{n}_\infty - \int_S dS \left(\Theta_0 \mathbf{q}_1 - \Theta_1 \mathbf{q}_0 + \frac{1}{2}\Theta_0^2 \mathbf{u}\right).\mathbf{n} = 0 \tag{1.62}$$

The integral over the surface of the object is considerably simplified, because we know that $\Theta_0 = 1$, $\Theta_1 = 0$ and $\mathbf{u} = 0$ on the surface of the object. Therefore, the above equation reduces to,

$$\int_{S_\infty} dS_\infty \left(\Theta_0 \mathbf{q}_1 - \Theta_1 \mathbf{q}_0 + \frac{1}{2}\Theta_0^2 \mathbf{u}\right).\mathbf{n}_\infty - \int_S dS \mathbf{q}_1.\mathbf{n} = 0 \tag{1.63}$$

The last term on the left side of the above equation is the heat generated by the object of arbitrary shape. Therefore, this term can be evaluated if the other integrals over the surface S_∞ at a large distance from the object are known. Without loss of generality, we can assume that this surface is a spherical surface, and the outward unit normal to this surface is in the radial direction.

In order to evaluate the integrals in equation 1.62, it is necessary to have expressions for Θ_0 and Θ_1 in the limit $r^* \to \infty$.

1. Consider the integral,

$$\frac{1}{2}\int_{S_\infty} dS_\infty \Theta_0^2 \mathbf{u}.\mathbf{n}_\infty \tag{1.64}$$

The temperature $\Theta_0 \to (\mathrm{Nu}_0/2r^*)$ in the limit $r^* \to 0\infty$ from equation 1.50, while the velocity $\mathbf{u}^* \to \cos(\theta)\mathbf{e}_r + \sin(\theta)\mathbf{e}_\theta$. Since the unit normal over the surface of a sphere is $\mathbf{n} = \mathbf{e}_r$, the above integral evaluated over the surface S_∞ is,

$$\begin{aligned} \frac{1}{2}\int_{S_\infty} dS_\infty \Theta_0^2 \mathbf{u}.\mathbf{n}_\infty &= \int r^{*2}\sin(\theta) d\theta d\phi \left(\frac{\mathrm{Nu}_0}{2r^{*2}}\right)\cos\theta \\ &= 0 \end{aligned} \tag{1.65}$$

2. The integral

$$\int_{S_\infty} dS_\infty \Theta_0 \mathbf{q}_1.\mathbf{n}_\infty \tag{1.66}$$

is also zero in the limit $r^* \to \infty$. This is because the heat flux due to the leading approximation for Θ_1, equation 1.53, is zero, because the solution 1.53 is a constant. The next higher contribution to Θ_1 is proportional to $(1/r^*)$, and the heat flux due to this is proportional to $(1/r^{*2})$. The leading contribution to Θ_0 is proportional to $(1/r^*)$, and therefore the integrand in equation 1.65 is proportional to $(1/r^{*3})$. Since the surface area increases proportional to r^{*2}, the integral in equation 1.65 decreases proportional to $(1/r^*)$, and is zero in the limit $r^* \gg 1$.

3. Finally, the integral

$$\int_{S_\infty} dS_\infty \Theta_1 \mathbf{q}_0.\mathbf{n}_\infty \tag{1.67}$$

can be easily evaluated for the leading solutions equations 1.52 and 1.53. From equation 1.52, the leading order heat flux is given by,$\mathbf{q}_0 = -(K(T_0 - T_\infty)/R)(\mathrm{Nu}_0 S/2r^2)$. Using this, we get

$$\int_{S_\infty} dS_\infty \Theta_1 \mathbf{q}_0.\mathbf{n}_\infty = \frac{K(T_0 - T_\infty)S\mathrm{Nu}_0}{2R}\frac{\mathrm{Nu}_0 U_\infty S}{4\alpha R} \tag{1.68}$$

Therefore, the final expression for the heat flux from the surface is,

$$Q_1 = \int_S dS\mathbf{q}_1.\mathbf{n} = \tag{1.69}$$

The first correction to the Peclet number is, therefore,

1.2.1 Streaming past an object of arbitrary shape:

In the case of streaming past an object of arbitrary shape, we would expect the expansion for the Nusselt number, in the limit of small Peclet number, to be of the form,

$$\mathrm{Nu} = \mathrm{Nu}_0 + \mathrm{Pe}\mathrm{Nu}_1 + \ldots \tag{1.70}$$

While it is necessary to determine the leading order contribution to the dimensionless heat flux, Nu_0, by solving the heat conduction equation for the specific shape under consideration, it can be shown that the first correction Nu_1 is related to the leading order dimensionless heat flux by

$$\mathrm{Nu}_1 = \frac{1}{4}\mathrm{Nu}_0^2 \tag{1.71}$$

This is an example of diffusion dominated processes where results can be obtained just from a knowledge of the 'far field' temperature distribution which is only a function of the most slowly decaying component of the temperature field, without a knowledge of the temperature distribution in the vicinity of the object. The procedure is as follows.

The governing equation for the temperature field is given by equation **??**, while the boundary conditions, instead of **??**, are

$$\begin{aligned}\Theta &= 1 \text{ on } S \\ \Theta &\to 0 \text{ for} r^* \to \infty\end{aligned} \tag{1.72}$$

where S is the surface of the object. The expansion for the temperature field is given in equation **??**, and the equations for the leading order and first correction to the temperature field, written in terms of the heat fluxes $\mathbf{q}_0 = -\nabla^*\Theta_0$ and $\mathbf{q}_1 = -\nabla^*\Theta_1$, are

$$\begin{aligned}\nabla^*\mathbf{q}_0 &= 0 \\ \Theta_0 &= 1 \text{ on } S\end{aligned} \tag{1.73}$$

$$\begin{aligned}\nabla^*\mathbf{q}_1 &= -\mathbf{u}^*.\nabla\Theta_0 \\ \Theta_1 &= 0 \text{ on } S\end{aligned} \tag{1.74}$$

It should be noted that equation **??** applies only for $r^\dagger = (Per^*) \ll 1$, since we have shown in the previous section that there is a correction due to convective effects for $r^\dagger \sim 1$. The Nusselt number can be calculated once the equations for the flux are known,

$$\begin{aligned}\mathrm{Nu}_0 &= \frac{2R}{KS}\int_S dS\mathbf{q}_0.\mathbf{n} \\ \mathrm{Nu}_1 &= \frac{2R}{KS}\int_S dS\mathbf{q}_1.\mathbf{n}\end{aligned} \tag{1.75}$$

where $\mathbf{n}$ is the unit normal to the surface.

Equation **??** for the first correction to the temperature field can be rewritten, noting that $\nabla.\mathbf{u}^* = 0$, to obtain

$$\nabla^*.(\mathbf{q}_1 + \mathbf{u}^*\Theta_0) = 0 \tag{1.76}$$

It is convenient to multiply **??** by Θ_0, and rewrite the result in the form,

$$\nabla^*.\left(\Theta_0\mathbf{q}_1 - \Theta_1\mathbf{q}_0 + \frac{1}{2}\Theta_0^2\mathbf{u}\right) = 0 \tag{1.77}$$

Here, we have used the simplifiation

$$\begin{aligned}
\Theta_0 \nabla^*.\mathbf{q}_1 &= \nabla^*.(\Theta_0 \mathbf{q}_1) - \mathbf{q}_1.\nabla^* \Theta_0 \\
&= \nabla^*.(\Theta_0 \mathbf{q}_1) - (\nabla^* \Theta_1).(\nabla^* \Theta_0) \\
&= \nabla^*.(\Theta_0 \mathbf{q}_1) - \nabla^*.(\Theta_1 \nabla^* \Theta_0) + \Theta_1 \nabla^{*2} \Theta_0 \\
&= \nabla^*.(\Theta_0 \mathbf{q}_1) - \nabla^*.(\Theta_1 \nabla^* \Theta_0) \qquad (1.78)
\end{aligned}$$

The final equality follows from the leading order equation **??**, since $\nabla^* \Theta_0$ is zero.

Next, we use the divergence theorem to express equation **??** as the difference between two surface integrals, one at the surface of the object S, and the other at a surface S_∞ which is at a large distance from the object $r^* \to \infty$.

$$\int_{S_\infty} dS_\infty \left(\Theta_0 \mathbf{q}_1 - \Theta_1 \mathbf{q}_0 + \frac{1}{2} \Theta_0^2 \mathbf{u} \right).\mathbf{n}_\infty - \int_S dS \left(\Theta_0 \mathbf{q}_1 - \Theta_1 \mathbf{q}_0 + \frac{1}{2} \Theta_0^2 \mathbf{u} \right).\mathbf{n} = 0 \qquad (1.79)$$

The integral over the surface of the object is considerably simplified, because we know that $\Theta_0 = 1$, $\Theta_1 = 0$ and $\mathbf{u} = 0$ on the surface of the object. Therefore, the above equation reduces to,

$$\int_{S_\infty} dS_\infty \left(\Theta_0 \mathbf{q}_1 - \Theta_1 \mathbf{q}_0 + \frac{1}{2} \Theta_0^2 \mathbf{u} \right).\mathbf{n}_\infty - \int_S dS \mathbf{q}_1.\mathbf{n} = 0 \qquad (1.80)$$

The last term on the left side of the above equation is the heat generated by the object of arbitrary shape. Therefore, this term can be evaluated if the other integrals over the surface S_∞ at a large distance from the object are known.

In order to evaluate the integrals in equation **??**, it is necessary to have expressions for Θ_0 and Θ_1 in the limit $r^* \to \infty$. First, we note that while determining the outer solution for the temperature field for $r^\dagger = (\mathrm{Pe} r^*) \sim 1$ from equation **??** to provide the solution **??**, we did not solve for any specific shape of the particle. We only used the fact that in the limit $r^\dagger \sim 1$, the fluid velocity was given by the free stream velocity $\mathbf{u}^* = \mathbf{e}_r \cos(\theta) - \mathbf{e}_\theta \sin(\theta)$. This approximation for the velocity field is valid independent of the shape of the object, and therefore the solution for the leading order temperature in this region is also valid independent of the shape of the object, subject only to the condition that $\Theta_0 = 1$ on the surface of the object.

The first correction to the temperature field Θ_1 was obtained by matching the solution of equation **??** with the outer solution. In this solution, the terms which are independent of r^* were obtained by matching the solution for Θ_1 with the leading order outer solutions in the limit $r^* \to \infty$. Therefore, these solutions are also independent of the shape of the object in the limit $r^* \to \infty$, and are applicable for any shape subject to the condition $\Theta_0 = 1$ and $\Theta_1 = 0$ on the surface of the object. Therefore, the solution for the first correction to the temperature field, in the limit $r^* \gg 1$, are

$$\Theta_1 = \frac{1}{2}(\cos(\theta) - 1) \tag{1.81}$$

We next turn to the evaluation of the integrals in equation **??**.

1. Consider the integral,

$$\frac{1}{2}\int_{S_\infty} dS_\infty \Theta_0^2 \mathbf{u}.\mathbf{n}_\infty \tag{1.82}$$

The temperature $\Theta_0 \to C/r^*)$ in the limit $r^* \to 0\infty$, where C is a constant, while the velocity $\mathbf{u}^* \to \cos(\theta)\mathbf{e}_r + \sin(\theta)\mathbf{e}_\theta$. Since the unit normal over the surface of a sphere is $\mathbf{n} = \mathbf{e}_r$, the above integral evaluated over the surface S_∞ is,

$$\begin{aligned} \frac{1}{2}\int_{S_\infty} dS_\infty \Theta_0^2 \mathbf{u}.\mathbf{n}_\infty &= \int r^{*2}\sin(\theta) d\theta d\phi \left(\frac{C}{r^{*2}}\right)\cos\theta \\ &= 0 \end{aligned} \tag{1.83}$$

2. The integral

$$\int_{S_\infty} dS_\infty \Theta_0 \mathbf{q}_1.\mathbf{n}_\infty \tag{1.84}$$

can be evaluated using the approximation $\Theta_0 = (1/r^*)$ and $\mathbf{q}_1 = \nabla^*\Theta_1 = (-\sin(\theta)/r^*)\mathbf{e}_\theta$ in the limit $r^* \to \infty$. It is easy to see that the above integral is identically zero, since the leading approximation for $\mathbf{q}_1$ is orthogonal to the unit normal to the surface, which is along the radial direction.

3. Finally, the integral

$$\int_{S_\infty} dS_\infty \Theta_1 \mathbf{q}_0.\mathbf{n}_\infty \tag{1.85}$$

can be easily evaluated for $\mathbf{q}_0 = (1/r^{*2})$ and $\Theta_1 = (\cos(\theta) - 1)/2$. Using this, we get

$$\int_{S_\infty} dS_\infty \Theta_1 \mathbf{q}_0.\mathbf{n}_\infty = -1 \tag{1.86}$$

Therefore, the final expression for the heat flux from the surface is,

$$\int_S dS\mathbf{q}_1.\mathbf{n} = 1 \tag{1.87}$$

1.2.2 Creeping flow past flat plate:

We start with the simplest geometry which could be considered, which is the flow part a flat plate. The configuration consists of a flat plate in the $x - z$ plane, bounding a fluid which occupies the half-space $y > 0$, as shown in figure **??**. The plate has a hot surface with temperature T_0 in the interval $0 < x < L$, while the rest of the plate is at the temperature $T_\infty < T_0$. The velocity of the fluid in the x direction is given by $u_x = \dot{\gamma}y$, where $\dot{\gamma}$ is the strain rate, and the fluid velocity at the surface of the plate is zero. The fluid upstream of the plate is at a temperature T_∞, and the surface of the plate is maintained at $T = T_\infty$ for $x < 0$, while the surface of the plate for $x > 0$ is heated to $T = T_0$. The fluid temperature at a large distance from the plate, $y \to \infty$, is given by $T = T_\infty$. We would like to find the heat flux from the surface of the plate in the limit where convective effects dominate over diffusive effects. The Peclet number for this case can be defined as $\mathrm{Pe} = (UL/\alpha) = (\dot{\gamma}L^2/\alpha)$, since the appropriate velocity scale $U = \dot{\gamma}L$, and we are interested in the limit of high Peclet number, $\mathrm{Pe} \gg 1$.

We first define the scaled temperature $\Theta = (T - T_\infty)/(T_0 - T_\infty)$. The convection-diffusion equation for Θ in the present geometry is,

$$\dot{\gamma}y\frac{\partial\Theta}{\partial x} = \alpha\left(\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2}\right) \tag{1.88}$$

The boundary conditions are,

$$\begin{aligned}
\Theta &= 0\ \mathrm{at} y = 0\ x < 0 \\
\Theta &= 1\ \mathrm{at} y = 0\ 0 < x < L \\
\Theta &= 0\ \mathrm{for} y \to \infty
\end{aligned} \tag{1.89}$$

The natural length scale for the x coordinate is the length of the plate, L, and so we define $x^* = (x/L)$. The length scale for the y coordinate is not so obvious, and so we define the scaled y coordinate as $(y^* = (y/l))$, where l is a length scale for the variation of the temperature perpendicular to the surface, which will be chosen so that there is a balance between the convective and diffusive terms. This is inserted into the convection-diffusion equation to obtain, and the result is multiplied by (l^2/α) to make all terms dimensionless, we obtain,

$$\left(\frac{\dot{\gamma}l^3}{\alpha L}\frac{\partial\Theta}{\partial x^*}\right) = \left(\frac{l^2}{L^2}\frac{\partial\Theta}{\partial x^{*2}} + \frac{\partial\Theta}{\partial y^{*2}}\right) \tag{1.90}$$

The factor $(\dot{\gamma}l^3/\alpha L)$ on the left side of the above equation can be recast as $\mathrm{Pe}(l/L)^3$, since the Peclet number for this problem is $\mathrm{Pe} = (\dot{\gamma}L^2/\alpha)$. There is a balance between the convection term on the left and the second diffusion term on the right only if $\mathrm{Pe}(l/L)^3 \sim 1$, or $l \sim \mathrm{Pe}^{-1/3}L$. In this case, the factor (l^2/L^2) multiplying the first term on the right is $\mathrm{Pe}^{-2/3}$, which is small in the limit $\mathrm{Pe} \gg 1$. Therefore, a balance is achieved between the term on the left and the second convective term on the right. Without loss of generality, we set $l = \mathrm{Pe}^{-1/3}L$, and the convection-diffusion equation **??** becomes,

$$\begin{aligned}\left(y^*\frac{\partial\Theta}{\partial x^*}\right) &= \left(\mathrm{Pe}^{-2/3}\frac{\partial\Theta}{\partial x^{*2}} + \frac{\partial\Theta}{\partial y^{*2}}\right)\\ &\simeq \frac{\partial\Theta}{\partial y^{*2}}\end{aligned} \tag{1.91}$$

Equation **??** can be solved using an appropriate method, such as separation of variables. However, an important simplification can be made if we recognise that when convective effects are dominant, the temperature at a location is determined primarily by the convection of fluid from regions upstream of this location, and is not affected by the temperature distribution downstream of this location. For example, at a point (x, y) for which the x coordinate is in the heated region $(0 < x < L)$, the temperature will depend only on the distance of x from the beginning of the heated plate and the distance y from the surface, but not on the total length L of the heated region. Therefore, the temperature distribution for $(0 < x < L)$ and $y < 0$ does not depend on the total distance of the heated region L. For this region, we solve the partial differential equation,

$$\dot{\gamma}y\frac{\partial\Theta}{\partial x} = \alpha\left(\frac{\partial^2\Theta}{\partial y^2}\right) \tag{1.92}$$

In the above equation, diffusion in the x direction along the surface has been neglected, since we have already established in equation **??** that the diffusion in the x direction is small compared to that in the cross-stream y direction. The boundary conditions are,

$$\begin{aligned}
\Theta &= 0 \,\mathrm{at} y = 0 \; x < 0 \\
\Theta &= 1 \,\mathrm{at} y = 0 \; x > 0 \\
\Theta &= 0 \,\mathrm{for} y \to \infty
\end{aligned} \tag{1.93}$$

Equation **??** can be solved using the similarity solution technique, since there are, now, no length scales in the problem. We define a similarity variable

$$\eta = (y/g(x)), \tag{1.94}$$

where $g(x)$ is defined in such a way that after transforming the independent variables from (x, y) to η, the resulting equation is a function of η alone. The physical motivation for introducing this transformation is the same as that used for rescaling the y coordinate in the high Peclet number limit in equation **??**. In the absence of diffusion, we would expect the temperature field to be a constant along fluid streamlines, which are straight lines in the flow direction in the present configuration. However, the presence of diffusion, there is a small distance shown to be of $O(\mathrm{Pe}^{-1/3}L)$ from the surface, where the temperature is modified due to diffusive effects, where $\mathrm{Pe} = (\dot{\gamma}L^2/\alpha)$. Further, we assumed in the previous paragraph, that the boundary layer solution does not depend on the total distance L of the heated plate, but only on the upstream distance x from the observation point. Therefore, we would expect the distance of propagation, $g(x)$ to scale as $(\mathrm{Pe}_x^{-1/3}x)$, where $\mathrm{Pe}_x = (\dot{\gamma}x^2/\alpha)$. Further calculations show that this is, indeed, the case.

The spatial derivatives of the temperature field, expressed in terms of η, are,

$$\begin{aligned}
\frac{\partial \Theta}{\partial y} &= \frac{1}{g(x)}\frac{d\Theta}{d\eta} \\
\frac{\partial^2 \Theta}{\partial y^2} &= \frac{1}{g(x)^2}\frac{d^2\Theta}{d\eta^2} \\
\frac{\partial \Theta}{\partial x} &= -\frac{y}{g(x)^2}\frac{dg}{dx}\frac{d\Theta}{d\eta}
\end{aligned} \tag{1.95}$$

This is inserted into the convection-diffusion equation, **??**, to obtain,

$$-\dot{\gamma}y\frac{dg}{dx}\frac{y}{g(x)^2}\frac{d\Theta}{d\eta} = \frac{\alpha}{g(x)^2}\frac{d^2\Theta}{d\eta^2} \tag{1.96}$$

The boundary conditions **??**, expressed in terms of the similarity variable η, are,

$$\begin{aligned}\Theta &= 0\ \text{at} y = 0\ x = 0 \rightarrow \eta \rightarrow \infty 0 \\ \Theta &= 1\ \text{at} y = 0\ x > 0 \rightarrow \eta = 0 \\ \Theta &= 0\ \text{for} y \rightarrow \infty \rightarrow \eta \rightarrow \infty \end{aligned} \tag{1.97}$$

We express y in terms of $g(x)$ and η, and multiply throughout by $(g(x)^2/\alpha)$, to obtain,

$$-\left(\frac{\dot{\gamma} g(x)^2}{\alpha}\frac{dg}{dx}\right)\eta^2\frac{d\Theta}{d\eta} = \frac{d^2\Theta}{d\eta^2} \tag{1.98}$$

In the above equation, the left side is a function both of x and η, whereas the right side is only a function of η. A similarity solution can be obtained only if the function in brackets on the left side of equation **??** is independent of x,

$$\frac{\dot{\gamma} g(x)^2}{\alpha}\frac{dg}{dx} = \text{Constant} \tag{1.99}$$

The exact value of the constant in the above equation does not change the nature of the solution, but only serves to alter the scaling of the y coordinate in terms of the similarity variable η. Therefore, without loss of generality, we can set the value of the constant equal to 1, and solve the differential equation **??** to obtain,

$$g(x) = ((x\alpha/\dot{\gamma}) + C)^{1/3} \tag{1.100}$$

where C is a constant of integration. The constant of integration can be determined from the physical expectation that $g(x)$ is the distance over which energy diffuses in the y coordinate, as discussed previously. Since the surface is heated only in the domain $x > 0$, it is expected that $g(x)$, the distance for the diffusion of energy in the y coordinate, is identically zero at $x = 0$, and the appropriate boundary condition is,

$$g(x) = 0\ \text{at} x = 0 \tag{1.101}$$

Thus, the constant $C = 0$ in equation **??**, and the function $g(x)$ is given by,

$$g(x) = \left(\frac{x\alpha}{\dot{\gamma}}\right)^{1/3}$$

$$
\begin{aligned}
&= \left(\frac{\alpha}{x^2\dot{\gamma}}\right)^{1/3} x \\
&= \mathrm{Pe}_x^{-1/3} x \qquad (1.102)
\end{aligned}
$$

Thus, the boundary layer thickness $g(x) = \mathrm{Pe}_x^{-1/3} x$, as anticipated earlier, where $\mathrm{Pe}_x = (x^2\dot{\gamma}/\alpha)$ is the Peclet number based on the downstream distance from the beginning of the heated surface.

Equation **??** for the temperature field can be easily solved, to obtain,

$$
\Theta = C_1 \int_0^{\eta} d\eta' \exp\left(-\eta'^3/3\right) + C_2 \qquad (1.103)
$$

wher C_1 and C_2 are constants of integration. The solutions for these two constants, by application of the boundary conditions **??**, are,

$$
\begin{aligned}
C_1 &= -\frac{1}{\int_0^{\infty} d\eta' \exp\left(-\eta'^3/3\right)} \\
C_2 &= 1 \qquad (1.104)
\end{aligned}
$$

Thus, the final solution for the temperature field is,

$$
\Theta = 1 - \frac{\int_0^{\eta} d\eta' \exp\left(-\eta'^3/3\right)}{\int_0^{\infty} d\eta' \exp\left(-\eta'^3/3\right)} \qquad (1.105)
$$

The heat flux from the surface in the y direction, q_y, can now be calculated,

$$
\begin{aligned}
q_y &= -K \left.\frac{\partial T}{\partial y}\right|_{y=0} \\
&= -\frac{K(T_0 - T_\infty)}{g(x)} \left.\frac{d\Theta}{d\eta}\right|_{\eta=0} \\
&= \frac{K(T_0 - T_\infty)}{(x\alpha/\dot{\gamma})^{1/3}} \frac{1}{\int_0^{\infty} d\eta' \exp\left(-\eta'^3/3\right)} \\
&= \frac{K(T_0 - T_\infty)}{(x\alpha/\dot{\gamma})^{1/3}} \frac{\Gamma(1/3)}{3^{2/3}} \qquad (1.106)
\end{aligned}
$$

The total heat transferred from a heated surface of length L, per unit length in the z direction, Q, can be determined by integrating the heat flux per unit length over the x coordinate,

$$\begin{aligned} Q &= \int_0^L dx q_y \\ &= \frac{K(T_0 - T_\infty)L^{2/3}}{(\alpha/\dot{\gamma})^{1/3}} \frac{\Gamma(1/3)3^{1/3}}{2} \\ &= K(T_0 - T_\infty)\mathrm{Pe}^{1/3}\frac{\Gamma(1/3)3^{1/3}}{2} \end{aligned} \tag{1.107}$$

where $Pe = (\dot{\gamma}L^2/\alpha)$. From this expression, the Nusselt number can be calculated as,

$$Nu = \frac{2Q}{K(T_0 - T_\infty)} = 3^{1/3}\Gamma(1/3)\mathrm{Pe}^{1/3} \tag{1.108}$$

1.2.3 Diffusion from a solid particle

In laminar flow, the velocity field around a particle is given by,

$$u_r = U\cos(\theta)\left(1 - \frac{3}{2r^*} + \frac{1}{2r^{*3}}\right) \tag{1.109}$$

$$u_\theta = -U\sin(\theta)\left(1 - \frac{3}{4r^*} - \frac{1}{rr^{*3}}\right) \tag{1.110}$$

We define the 'inner coordinate' $y^* = (r - R)/R = ((1 - r^*)/\epsilon)$, so that $y^* \sim 1$ in the inner region where $(1 - r^*) \sim \epsilon$. With this rescaling, we can express the equations for the velocity field in the inner region as,

$$u_r^* = \left(1 - \frac{3}{2(1 + \epsilon y^*)} + \frac{1}{2(1 + \epsilon y^*)^3}\right)\cos(\theta) \tag{1.111}$$

$$u_\theta^* = -\left(1 - \frac{3}{4(1 + \epsilon y^*)} - \frac{1}{4(1 + \epsilon y^*)^3}\right)\sin(\theta) \tag{1.112}$$

Since the parameter ϵ is small and we are interested in the region $y^* \sim 1$, the expressions for the components of the velocity, **??** and **??**, can be simplified by retaining just the leading order terms in an expansion in ϵ,

$$u_r^* = \frac{3}{2}\epsilon^2 y^{*2}\cos(\theta) \tag{1.113}$$

$$u_\theta^* = -\frac{3}{2}\epsilon y^* \sin(\theta) \tag{1.114}$$

These expressions for the components of the velocity are inserted into the convection-diffusion equation, **??**, and the radial coordinate is written in terms of the y^* using the transformation $r^* = (1 + \epsilon y^*)$, to obtain,

$$\begin{aligned} \mathrm{Pe}\left(\frac{3}{2}\epsilon^2 y^{*2}\cos(\theta)\frac{1}{\epsilon}\frac{\partial\Theta}{\partial y^*} - \frac{3}{2}y^*\epsilon\sin(\theta)\frac{1}{(1+\epsilon y^*)}\frac{\partial\Theta}{\partial\theta}\right) \\ = \frac{1}{(1+\epsilon y^*)^2}\frac{1}{\epsilon}\frac{\partial}{\partial y^*}\left((1+\epsilon y^*)^2\frac{1}{\epsilon}\frac{\partial\Theta}{\partial y^*}\right) \\ +\frac{1}{(1+\epsilon y^*)^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Theta}{\partial\theta}\right) \end{aligned} \tag{1.115}$$

It is apparent that the term on the left side is proportional to $\mathrm{Pe}\epsilon$ in the limit $\epsilon \to 0$, while the largest term on the right, which is the first term on the right, is proportional to $(1/\epsilon^2)$. A balance is achieved only for $\epsilon \sim \mathrm{Pe}^{-1/3}$, so that the thickness of the boundary layer decreases proportional to $R\mathrm{Pe}^{-1/3}$ in the limit $\mathrm{Pe} \gg 1$. If we substitute $\epsilon = \mathrm{Pe}^{-1/3}$ in equation **??**, use the asymptotic expansion **??** for Θ_0, and retain only the largest terms in the expansion, we get,

$$\frac{3}{2}y^{*2}\cos(\theta)\frac{\partial\Theta_0}{\partial y^*} - \frac{3}{2}y^*\sin(\theta)\frac{\partial\Theta_0}{\partial\theta} = \frac{\partial^2\Theta_0}{y^{*2}} \tag{1.116}$$

This is the 'boundary layer' equation for the thermal boundary layer at the surface of the sphere.

In order to solve the equation, we note that there are boundary conditions on the surface of the sphere ($y^* = 0$) and in the limit $y^* \to \infty$. Since there is no length scale in the problem,we can examine a solution in terms of a similarity variable, η,

$$\Theta_0 = \Theta_0(\eta) \tag{1.117}$$

where the similarity variable is defined as,

$$\eta = (y^*/h(\cos(\theta)) \tag{1.118}$$

The derivatives of Θ_0 can then be expressed only in terms of the variable η,

$$\frac{\partial\Theta_0}{\partial y^*} = \frac{1}{h}\frac{\partial\Theta_0}{\partial\eta} \tag{1.119}$$

$$\frac{\partial^2 \Theta_0}{\partial y^{*2}} = \frac{1}{h^2}\frac{\partial^2 \Theta_0}{\partial \eta^2} \tag{1.120}$$

$$\frac{\partial \Theta_0}{\partial \theta} = -\frac{y^*}{h^2}\frac{dh}{d\theta}\frac{\partial \Theta_0}{\partial \eta} \tag{1.121}$$

Substituting this into the convection-diffusion equation **??**, we get,

$$\frac{d^2\Theta_0}{d\eta^2} - \frac{3}{2}\eta^2 \frac{d\Theta_0}{d\eta}\left(h^3\cos{(\theta)} + \sin{(\theta)} h^2 \frac{dh}{d\theta}\right) = 0 \tag{1.122}$$

The above equation has a similarity solution only is the term in the braces, $(h^3 + \sin{(\theta)}^2 h^2 (dh/d\theta))$, is a constant. The numerical value of this constant can be chosen arbitrarily, but this constant has to be negative, so that the solution for the temperature is finite in the limit $\eta \to \infty$. The simplest choice is,

$$\left(\sin{(\theta)} h^2 \frac{dh}{d\theta} + h^3 \cos{(\theta)}\right) = -1 \tag{1.123}$$

This equation can be solved by first obtaining the homogeneous solution h_h, which is the solution of the above equation with the right side set equal to zero, and then using the product rule to find the general solution. It can easily be verified that the solution of the homogeneous equation, upto a constant prefactor, is,

$$h_h =$$

The constant C has to be determined from the boundary condition in the θ coordinate. This constant is determined from the condition at $\theta = \Pi$, which is the upstream stagnation point on the sphere. At this point, the derivative of h with respect to θ is well defined, along all approaches, only if $(dh/d\theta) = 0$. If we apply this condition, we find that the constant $C = 0$, so that the solution for $h(\theta)$ is,

$$h(\theta) = \frac{1}{\sin{(\theta)}}\left(\frac{Pi - 2\theta + \sin{(2\theta)}}{2}\right)^{1/3} \tag{1.125}$$

With this substitution, the equation **??** reduces to,

$$\frac{d^2\Theta_0}{d\eta^2} + \frac{3\eta^2}{2}\frac{d\Theta_0}{d\eta} = 0 \tag{1.126}$$

This equation can be integrated once by parts to obtain,

$$\frac{d\Theta_0}{dt} = A \exp\left(-\eta^3/2\right) \tag{1.127}$$

After integrating a second time, we obtain the most general solution,

$$\Theta_0 = A \int_0^\eta d\eta' \exp\left(-\eta'^3/2\right) + B \tag{1.128}$$

The boundary conditions require that in the limit $y^* \to \infty$, $\Theta_0 = 0$, while $\Theta_0 = 1$ at $y^* = 0$. With these, the general solution for the temperature field becomes,

$$\Theta_0 = 1 - \frac{\int_0^\eta d\eta' \exp\left(-\eta'^3/2\right)}{\int_0^\infty d\eta' \exp\left(-\eta'^3/2\right)} \tag{1.129}$$

The radial heat flux at the surface is given by,

$$\begin{aligned} q_r &= -\frac{K}{R}\frac{d\Theta_0}{dr^*}\bigg|_{r^*=1} \\ &= -\frac{K\mathrm{Pe}^{1/3}}{R}\frac{d\Theta_0}{dy^*}\bigg|_{y^*=0} \end{aligned}$$

$$-\frac{K\Pi^{1/3}}{Rh(\theta)}\frac{dT_0}{d\eta}\bigg|_{\eta=0}$$

$$= -\frac{KPe^{1/3}}{Rh(\theta)}\frac{1}{\int_0^\infty d\eta' \exp\left(-\eta'^3/2\right)} \tag{1.130}$$

This flux is a function of position θ on the surface, and so the total heat transferred from the surface to the fluid is given by,

$$Q = -\frac{KPe^{1/3}}{R}\frac{1}{\int_0^\infty d\eta' \exp\left(-\eta'^3\right)}R^2\int_0^{2\pi} d\phi \int_0^\pi \sin(\theta) d\theta \frac{1}{h(\theta)} \tag{1.131}$$

It is necessary to evaluate the two integrals in the above expression numerically, and the numerical expressions for the integrals are,

$$\frac{1}{\int_0^\infty d\eta' \exp\left(-\eta'^3/2\right)} = 2^{1/3}\Gamma\left(\frac{4}{3}\right) = 0.8930 \tag{1.132}$$

$$\int_0^\pi \sin(\theta) d\theta \frac{1}{h(\theta)} = 1.11546 \tag{1.133}$$

Inserting the numerical values of the above integrals, we find the final expression for the total heat transferred from the particle is

$$Q = 1.2491(2\pi R K \mathrm{Pe}^{1/3})(T_0 - T_\infty) \tag{1.134}$$

The Nusselt number is obtained by dividing this by

$$\begin{aligned} Nu &= \frac{2Q}{(4\pi R^2 K(T_0 - T_\infty)/R)} \\ &= 1.2491\mathrm{Pe}^{1/3} \end{aligned} \tag{1.135}$$

1.2.4 General considerations:

In this section, we examine the extent to which the previous results obtained for spherical particles and bubbles can be generalised to arbitrary shapes of particles, drops and bubbles.

First consider the case of the fluid flow past a solid particle in which the velocity field satisfies the no-slip condition at the surface. The characteristic length scale of the particle is L, and the fluid velocity at a large distance from the particle is U_∞ in the x direction, as shown in figure **??**. We know, from the previous discussions, that the concentration or temperature variations are restricted to a boundary layer of thickness ϵL at the surface, where ϵ is a small parameter to be determined from a balance between convection and diffusion. The variation of the fluid velocity field very near the surface can be inferred as follows. Consider a coordinate system at a point on the surface, where the coordinate normal to the surface is ζ and the coordinate tangential to the surface is ξ, as shown in figure **??**. The components of the mean velocity are required, by the no-slip condition, to be equal to zero at the surface $\zeta = 0$. Therefore, very near the surface for $\zeta \ll L$, it is expected that the two components of the velocity are linear in the coordinate ζ, i. e., $u_\xi = U_\infty A(\xi)(\zeta/L)$ and $u_\zeta = U_\infty B(\xi)(\zeta/L)$, where $A(\xi)$ and $B(\xi)$ are dimensionless functions of the tangential coordinate ξ. However, for an incompressible fluid, this velocity field does not satisfy the mass conservation condition for an incompressible fluid. The mass conservation condition, $\nabla.\mathbf{u} = 0$, written in terms of ξ and ζ near the surface is,

$$\frac{\partial u_\xi}{\partial \xi} + \frac{\partial u_\zeta}{\partial \zeta} = 0 \tag{1.136}$$

If we assume $u_\xi = A(\xi)\zeta$, the above equation requires that $u_\zeta = (-(dA(\xi)/d\xi)\zeta^2/2)$. Therefore, the fluid velocity in a thin layer near the surface can be approximated as,

$$\begin{aligned} u_\xi &= U_\infty A(\xi)\frac{\zeta}{L} \\ u_\zeta &= -U_\infty \frac{dA(\xi)}{d\xi}\frac{\zeta^2}{2L^2} \end{aligned} \tag{1.137}$$

The exact form of the function $A(\xi)$ depends on the details of the particle shape.

This velocity field is inserted into the convection-diffusion equation, to obtain,

$$U_\infty \left(A(\xi)\frac{\zeta}{L}\frac{\partial\Theta}{\partial\xi} - \frac{\zeta^2}{2L^2}A'(\xi)\frac{\partial\Theta}{\partial\zeta} \right) = \alpha \left(\frac{\partial^2\Theta}{\partial\xi^2} + \frac{\partial^2\Theta}{\partial\eta^2} \right) \tag{1.138}$$

where $A'(\xi) = (dA/d\xi)$. It is convenient to define a scaled tangential coordinate $\xi^* = (\xi/L)$, since L is the length scale for the variation of temperature in the flow direction. In the cross-stream direction, we postulate that there is a much smaller length ϵL over which convection and diffusion are of equal magnitude in the high Peclet number limit, where $\epsilon \ll 1$. Consequently, the dimensionless distance in normal to the surface is defined as $\zeta^* = (\zeta/\epsilon L)$. Inserting these into the convection diffusion equation, and multiplying throughout by $\epsilon^2 L^2$, we get,

$$\mathrm{Pe}\epsilon^3 \left(A(\xi^*)\zeta^*\frac{\partial\Theta}{\partial\xi^*} - \frac{\zeta^{*2}}{2}A'(\xi)\frac{\partial\Theta}{\partial\zeta^*} \right) = \left(\epsilon^2\frac{\partial^2\Theta}{\partial\xi^{*2}} + \frac{\partial^2\Theta}{\partial\zeta^{*2}} \right) \tag{1.139}$$

The above equation indicates that there is a balance between convection and diffusion only for $\epsilon \sim \mathrm{Pe}^{-1/3}$, and $\zeta^* = (\mathrm{Pe}^{1/3}\zeta/L)$. Thus, the boundary layer thickness is $O(\mathrm{Pe}^{-1/3})$ smaller than the particle length L for the general case of the flow past a solid surface. Without loss of generality, we set $\epsilon = \mathrm{Pe}^{-1/3}$, and neglect the first term on the right side of equation **??** since it is $O(\epsilon^2)$, to obtain the boundary layer equation,

$$\left(A(\xi^*)\zeta^*\frac{\partial\Theta}{\partial\xi^*} - \frac{\zeta^{*2}}{2}A'(\xi)\frac{\partial\Theta}{\partial\zeta^*} \right) = \frac{\partial^2\Theta}{\partial\zeta^{*2}} \tag{1.140}$$

To proceed further, we examine the possibility of a similarity solution, using the substitution,

$$\eta = \frac{\zeta^*}{h(\xi^*)} \tag{1.141}$$

The derivatives of Θ with respect to ξ^* and ζ^* can be expressed in terms of η using the chain rule,

$$\begin{aligned}\frac{\partial\Theta}{\partial\zeta^*} &= \frac{1}{h}\frac{\partial\Theta}{\partial\eta}\\ \frac{\partial^2\Theta}{\partial\zeta^{*2}} &= \frac{1}{h^2}\frac{\partial^2\Theta}{\partial\eta^2}\\ \frac{\partial\Theta}{\partial\xi} &= -\frac{\zeta^*}{h^2}\frac{dh}{d\xi}\frac{\partial\Theta}{\partial\eta}\end{aligned} \tag{1.142}$$

This is inserted into equation **??**, and the left and right sides are multiplied by h^2, to obtain,

$$\frac{\partial^2\Theta}{\partial\eta^2} + \eta^2\frac{\partial\Theta}{\partial\eta}\left(A(\xi)h(\xi)^2\frac{dh}{d\xi} + \frac{h(xi)^3}{2}\frac{dA}{d\xi}\right) = 0 \tag{1.143}$$

It is apparent that the above equation admits a similarity solution only if

$$\left(A(\xi)h(\xi)^2\frac{dh}{d\xi} + \frac{h(xi)^3}{2}\frac{dA}{d\xi}\right) = \text{Constant} \tag{1.144}$$

The above equation is can be rewritten as,

1.3 Creeping flow past flat plate:

We start with the simplest geometry which could be considered, which is the flow part a flat plate. The configuration consists of a flat plate in the $x - z$ plane, bounding a fluid which occupies the half-space $y > 0$, as shown in figure 1.3. The plate has a hot surface with temperature T_0 in the interval $0 < x < L$, while the rest of the plate is at the temperature $T_\infty < T_0$. The velocity of the fluid in the x direction is given by $u_x = \dot{\gamma}y$, where $\dot{\gamma}$ is the strain rate, and the fluid velocity at the surface of the plate is zero. The fluid upstream of the plate is at a temperature T_∞, and the surface of the plate is maintained at $T = T_\infty$ for $x < 0$, while the surface of the plate for $x > 0$ is heated to $T = T_0$. The fluid temperature at a large distance from the plate, $y \to \infty$, is given by $T = T_\infty$. We would like to find the heat flux from the surface of the plate in the limit where convective effects dominate over diffusive effects. The Peclet number for this case can be defined as

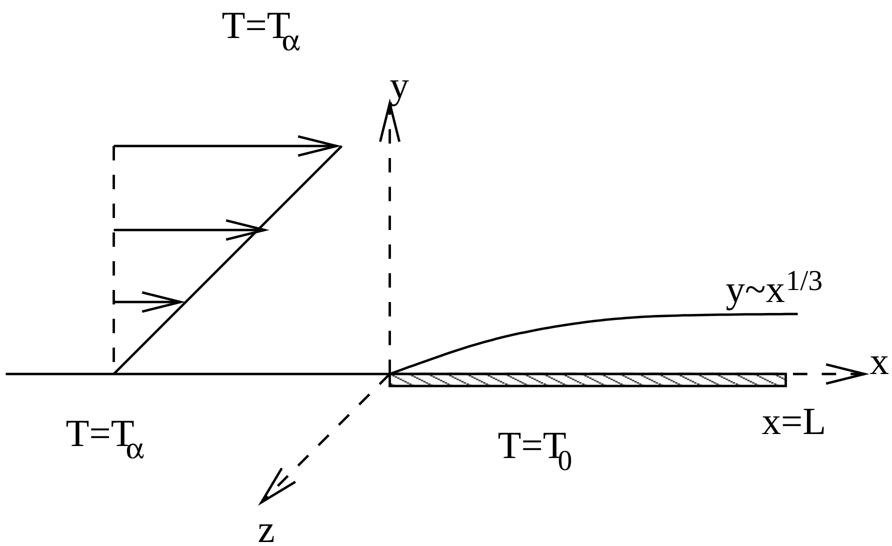

Figure 1.3: Forced convection due to the streaming flow past a heated surface of length L in the high Peclet number limit. The fluid velocity field is given by $u_x = \dot{\gamma} y$ and $u_y = 0$, the temperature on the heated section of the surface is maintained at T_0, and the temperature at a large distance from the surface is T_∞.

$\mathrm{Pe} = (UL/\alpha) = (\dot{\gamma}L^2/\alpha)$, since the appropriate velocity scale $U = \dot{\gamma}L$, and we are interested in the limit of high Peclet number, $\mathrm{Pe} \gg 1$.

We first define the scaled temperature $\Theta = (T - T_\infty)/(T_0 - T_\infty)$. The convection-diffusion equation for Θ in the present geometry is,

$$\mathbf{u}.\nabla\Theta = \alpha\nabla^2\Theta \tag{1.145}$$

The boundary conditions are,

$$\begin{aligned} \Theta &= 0 \,\mathrm{at} y = 0 \; x < 0 \\ \Theta &= 1 \,\mathrm{at} y = 0 \; 0 < x < L \\ \Theta &= 0 \,\mathrm{for} y \to \infty \end{aligned} \tag{1.146}$$

It is natural to scale all lengths by the length of the heated section of the flat plate, L, and velocities by $\dot{\gamma}L$, in the convection-diffusion equation 1.145. After scaling, we divide the equation by (α/L^2), to obtain the scaled convection-diffusion euation which can be written in component form as,

$$\mathrm{Pe} y^* \frac{\partial\Theta}{\partial x^*} = \frac{\partial^2\Theta}{\partial x^{*2}} + \frac{\partial^2\Theta}{\partial y^{*2}} \tag{1.147}$$

where the Peclet number is given by $\mathrm{Pe} = (\dot{\gamma}L^2/\alpha)$. We are considering the limit of large Peclet number, $\mathrm{Pe} \gg 1$. In this limit, we would naively expect that the right side of equation 1.147 can be neglected in comparison to the left side, and the leading order convection-diffusion equation can be approximated as,

$$y^* \frac{\partial\Theta}{\partial x^*} = 0 \tag{1.148}$$

This equation simply states that Θ is independent of the x^* coordinate, and Θ is only a function of y^*. However, we know that $\Theta = 0$ far upstream of the heated surface as $x^* \to -\infty$. Therefore, solution for equation 1.148 is, simply, $\Theta = 0$ everywhere in the flow. However, note that this solution *does not satisfy the boundary condition at* 1.146 at $y^* = 0$. So clearly, the solution of equation 1.148 is not a valid leading order solution for the original convection-diffusion equation 1.145.

The origin of the inconsistency can be traced to the fact that when we neglected the diffusion term on the right side in equation 1.147, we reduced the governing equation from a second-order partial differential equation to a first-order partial differential equation 1.148. It was necessary to specify two boundary conditions for the well-posedness of the original second-order partial differential equation 1.147, but it is possible to specify only one boundary condition for the first order differential equation 1.148. Physically, the inconsistency arises because we have assumed that the characteristic length in the cross-stream y^* direction is also L. In the high Peclet number limit, the rate of transport due to convection downstream is rapid compared to the rate of diffusion from the surface. Since material diffusing from the surface gets convected rapidly downstream, we would expect that the length to which the material diffuses perpendicular to the surface is small compared to the downstream distance. Therefore, we should use a much smaller distance l to scale the y coordinate. The length l is determined from the consistency condition that diffusion in the y direction is comparable to convection in the equation 1.147, so that the equation remains a second-order differential equation.

We define the scaled y coordinate as $(y^* = (y/l))$, where l is a length scale for the variation of the temperature perpendicular to the surface, which will be chosen so that there is a balance between the convective and diffusive terms. This is inserted into the convection-diffusion equation to obtain, and the result is multiplied by (l^2/α) to make all terms dimensionless, we obtain,

$$\left(\frac{\dot{\gamma} l^3}{\alpha L}\frac{\partial \Theta}{\partial x^*}\right) = \left(\frac{l^2}{L^2}\frac{\partial \Theta}{\partial x^{*2}} + \frac{\partial \Theta}{\partial y^{*2}}\right) \tag{1.149}$$

The factor $(\dot{\gamma} l^3/\alpha L)$ on the left side of the above equation can be recast as $\mathrm{Pe}(l/L)^3$, since the Peclet number for this problem is $\mathrm{Pe} = (\dot{\gamma} L^2/\alpha)$. There is a balance between the convection term on the left and the second diffusion term on the right only if $\mathrm{Pe}(l/L)^3 \sim 1$, or $l \sim \mathrm{Pe}^{-1/3} L$. In this case, the factor (l^2/L^2) multiplying the first term on the right is $\mathrm{Pe}^{-2/3}$, which is small in the limit $\mathrm{Pe} \gg 1$. Therefore, a balance is achieved between the term on the left and the second convective term on the right. Without loss of generality, we set $l = \mathrm{Pe}^{-1/3} L$, and the convection-diffusion equation 1.149 becomes,

$$\begin{aligned}\left(y^*\frac{\partial \Theta}{\partial x^*}\right) &= \left(\mathrm{Pe}^{-2/3}\frac{\partial \Theta}{\partial x^{*2}} + \frac{\partial \Theta}{\partial y^{*2}}\right) \\ &\simeq \frac{\partial \Theta}{\partial y^{*2}} \end{aligned} \tag{1.150}$$

Equation 1.150 can be solved using an appropriate method, such as separation of variables. However, an important simplification can be made if we recognise that when convective effects are dominant, the temperature at a location is determined primarily by the convection of fluid from regions upstream of this location, and is not affected by the temperature distribution downstream of this location. For example, at a point (x, y) for which the x coordinate is in the heated region $(0 < x < L)$, the temperature will depend only on the distance of x from the beginning of the heated plate and the distance y from the surface, but not on the total length L of the heated region. Therefore, the temperature distribution for $(0 < x < L)$ and $y < 0$ does not depend on the total distance of the heated region L. For this region, we solve the partial differential equation,

$$\dot{\gamma} y \frac{\partial \Theta}{\partial x} = \alpha \left(\frac{\partial^2 \Theta}{\partial y^2} \right) \tag{1.151}$$

In the above equation, diffusion in the x direction along the surface has been neglected, since we have already established in equation 1.150 that the diffusion in the x direction is small compared to that in the cross-stream y direction. The boundary conditions are,

$$\begin{aligned} \Theta &= 0 \,\mathrm{at} y = 0 \; x < 0 \\ \Theta &= 1 \,\mathrm{at} y = 0 \; x > 0 \\ \Theta &= 0 \,\mathrm{for} y \to \infty \end{aligned} \tag{1.152}$$

Equation 1.151 can be solved using the similarity solution technique, since there are, now, no length scales in the problem. We define a similarity variable

$$\eta = (y/g(x)), \tag{1.153}$$

where $g(x)$ is defined in such a way that after transforming the independent variables from (x, y) to η, the resulting equation is a function of η alone. The physical motivation for introducing this transformation is the same as that used for rescaling the y coordinate in the high Peclet number limit in equation 1.149. In the absence of diffusion, we would expect the temperature field to be a constant along fluid streamlines, which are straight lines in the flow direction in the present configuration. However, the presence of diffusion, there is a small distance shown to be of $O(\mathrm{Pe}^{-1/3} L)$ from the surface, where the temperature is modified due to diffusive effects, where $\mathrm{Pe} = (\dot{\gamma} L^2/\alpha)$. Further, we assumed in the previous paragraph, that the boundary layer solution does not depend on the total distance L of the heated plate, but only on the upstream distance x from the observation point. Therefore, we would expect the distance of propagation, $g(x)$ to scale as $(\mathrm{Pe}_x^{-1/3} x)$, where $\mathrm{Pe}_x = (\dot{\gamma} x^2/\alpha)$. Further calculations show that this is, indeed, the case.

The spatial derivatives of the temperature field, expressed in terms of η, are,

$$\begin{aligned}
\frac{\partial \Theta}{\partial y} &= \frac{1}{g(x)} \frac{d\Theta}{d\eta} \\
\frac{\partial^2 \Theta}{\partial y^2} &= \frac{1}{g(x)^2} \frac{d^2\Theta}{d\eta^2} \\
\frac{\partial \Theta}{\partial x} &= -\frac{y}{g(x)^2} \frac{dg}{dx} \frac{d\Theta}{d\eta}
\end{aligned} \tag{1.154}$$

This is inserted into the convection-diffusion equation, 1.151, to obtain,

$$-\dot{\gamma} y \frac{dg}{dx} \frac{y}{g(x)^2} \frac{d\Theta}{d\eta} = \frac{\alpha}{g(x)^2} \frac{d^2\Theta}{d\eta^2} \tag{1.155}$$

The boundary conditions 1.156, expressed in terms of the similarity variable η, are,

$$\begin{aligned}
\Theta &= 0 \,\mathrm{at} y = 0 \; x = 0 \rightarrow \eta \rightarrow \infty 0 \\
\Theta &= 1 \,\mathrm{at} y = 0 \; x > 0 \rightarrow \eta = 0 \\
\Theta &= 0 \,\mathrm{for} y \rightarrow \infty \rightarrow \eta \rightarrow \infty
\end{aligned} \tag{1.156}$$

We express y in terms of $g(x)$ and η, and multiply throughout by $(g(x)^2/\alpha)$, to obtain,

$$-\left(\frac{\dot{\gamma} g(x)^2}{\alpha} \frac{dg}{dx} \right) \eta^2 \frac{d\Theta}{d\eta} = \frac{d^2\Theta}{d\eta^2} \tag{1.157}$$

In the above equation, the left side is a function both of x and η, whereas the right side is only a function of η. A similarity solution can be obtained only if the function in brackets on the left side of equation 1.157 is independent of x,

$$\frac{\dot{\gamma} g(x)^2}{\alpha} \frac{dg}{dx} = \mathrm{Constant} \tag{1.158}$$

The exact value of the constant in the above equation does not change the nature of the solution, but only serves to alter the scaling of the y coordinate in terms of the similarity variable η. Therefore, without loss of generality, we can set the value of the constant equal to 1, and solve the differential equation 1.158 to obtain,

$$g(x) = ((x\alpha/\dot{\gamma}) + C)^{1/3} \tag{1.159}$$

where C is a constant of integration. The constant of integration can be determined from the physical expectation that $g(x)$ is the distance over which energy diffuses in the y coordinate, as discussed previously. Since the surface is heated only in the domain $x > 0$, it is expected that $g(x)$, the distance for the diffusion of energy in the y coordinate, is identically zero at $x = 0$, and the appropriate boundary condition is,

$$g(x) = 0 \text{ at} x = 0 \tag{1.160}$$

Thus, the constant $C = 0$ in equation 1.159, and the function $g(x)$ is given by,

$$\begin{aligned} g(x) &= \left(\frac{x\alpha}{\dot{\gamma}}\right)^{1/3} \\ &= \left(\frac{\alpha}{x^2\dot{\gamma}}\right)^{1/3} x \\ &= \mathrm{Pe}_x^{-1/3} x \end{aligned} \tag{1.161}$$

Thus, the boundary layer thickness $g(x) = \mathrm{Pe}_x^{-1/3} x$, as anticipated earlier, where $\mathrm{Pe}_x = (x^2\dot{\gamma}/\alpha)$ is the Peclet number based on the downstream distance from the beginning of the heated surface.

Equation 1.157 for the temperature field can be easily solved, to obtain,

$$\Theta = C_1 \int_0^\eta d\eta' \exp(-\eta'^3/3) + C_2 \tag{1.162}$$

wher C_1 and C_2 are constants of integration. The solutions for these two constants, by application of the boundary conditions 1.156, are,

$$\begin{aligned} C_1 &= -\frac{1}{\int_0^\infty d\eta' \exp(-\eta'^3/3)} \\ C_2 &= 1 \end{aligned} \tag{1.163}$$

Thus, the final solution for the temperature field is,

$$\Theta = 1 - \frac{\int_0^\eta d\eta' \exp(-\eta'^3/3)}{\int_0^\infty d\eta' \exp(-\eta'^3/3)} \tag{1.164}$$

The heat flux from the surface in the y direction, q_y, can now be calculated,

$$\begin{aligned} q_y &= -K \left.\frac{\partial T}{\partial y}\right|_{y=0} \\ &= -\frac{K(T_0 - T_\infty)}{g(x)} \left.\frac{d\Theta}{d\eta}\right|_{\eta=0} \\ &= \frac{K(T_0 - T_\infty)}{(x\alpha/\dot{\gamma})^{1/3}} \frac{1}{\int_0^\infty d\eta' \exp{(-\eta'^3/3)}} \\ &= \frac{K(T_0 - T_\infty)}{(x\alpha/\dot{\gamma})^{1/3}} \frac{\Gamma(1/3)}{3^{2/3}} \end{aligned} \tag{1.165}$$

The total heat transferred from a heated surface of length L, per unit length in the z direction, Q, can be determined by integrating the heat flux per unit length over the x coordinate,

$$\begin{aligned} Q &= \int_0^L dx q_y \\ &= \frac{K(T_0 - T_\infty)L^{2/3}}{(\alpha/\dot{\gamma})^{1/3}} \frac{\Gamma(1/3)3^{1/3}}{2} \\ &= K(T_0 - T_\infty)\mathrm{Pe}^{1/3} \frac{\Gamma(1/3)3^{1/3}}{2} \end{aligned} \tag{1.166}$$

where $Pe = (\dot{\gamma}L^2/\alpha)$. From this expression, the Nusselt number can be calculated as,

$$Nu = \frac{2Q}{K(T_0 - T_\infty)} = 3^{1/3}\Gamma(1/3)\mathrm{Pe}^{1/3} \tag{1.167}$$

It is useful to review some of the general features of the solution for the high Peclet number flow past a flat plate, since these are common to other problems of high Peclet number flow past solid surfaces. We start with the scaled convection-diffusion equation, 1.147, and the boundary conditions on the surface of the sphere are given by 1.146. The limit of high Peclet number is considered, in which convective effects are expected to be dominant compared to diffusive effects. The leading order equation for the temperature field is,

$$\mathbf{u}^*.\nabla^*\Theta_0 = 0 \tag{1.168}$$

The above equation just states that the temperature does not vary along fluid streamlines. Since the boundary condition requires that $\Theta_0 \to 0$ for $r^* \to \infty$, the only possible solution of the above equation is $\Theta_0 = 0$ everywhere in the flow. However, with this solution, it is not possible to satisfy the boundary condition $\Theta_0 = 1$ at $r^* = 1$. Clearly, this is an unsatisfactory situation, and this arises from two related causes, one physical and one mathematical.

1. The mathematical reason is that since we neglected the diffusive term in equation 1.147, the equation was converted from a second order differential equation to a first order differential equation. While it was possible to satisfy boundary conditions on two surfaces for the original second order differential equation, it is only possible to satisfy one boundary condition in the limit $x^* \to \infty$ in the modified equation 1.148, and it is not possible to satisfy the boundary condition on the surface.

2. This mathematical reason is related to the physical reason that convection transports mass and energy only in the direction of flow. Since there is no flow in the direction normal to the surface of the sphere, there can be no convective transport of energy in the direction normal to the surface of the sphere. Therefore, any flux from the surface of the sphere has to occur due to diffusion alone. Since equation 1.148 neglects the diffusive flux even at the surface of the sphere, we are not able to satisfy the boundary condition at the surface of the sphere.

From the above discussion, it is clear that diffusive transport has to be incorporated in order to determine the flux from the surface of the sphere. This can be done as follows.

1. Mathematically, we note that the convective part of the convection-diffusion equation 1.145 is proportional to the gradient of the temperature, while the diffusive part is proportional to the Laplacian. In scaling all lengths by the length of the heated section of the surface L, we have implicitly assumed that the gradient of the temperature scales as (Θ/L). In this case, the convective term in the convection-diffusion equation 1.147 is scales as $(\dot{\gamma}\Theta)$, while the diffusive term scales as $(\alpha\Theta/L^2)$. The ratio of the convective and diffusive terms is $(\dot{\gamma}L^2/\alpha) = \mathrm{Pe}$, which is large. However, if there is a variation in the temperature field over a much smaller than L, then the gradient in the temperature field at the surface would be much larger than that previously assumed, and the diffusive term in the convection-diffusion

equation (which is proportional to the second spatial derivative of the temperature) could become of the same magnitude as the convective term (which is proportional to the first spatial derivative). In this case, there could be a balance between convection and diffusion in the high Peclet number limit.

2. Physically, in the high Peclet number limit, any heat that diffuses out of the surface due to diffusion is convected rapidly downstream by the fluid flow. When convective effects are strong compared to diffusive effects, the rate at which heat is swept downstream is rapid compared to the rate at which it is convected form the surface. Therefore, on would expect the departure from $\Theta = 0$ to be confined to a thin layer at the surface of the sphere. However, the thickness of this region has to be determined by a balance between the convection and diffusion of energy close to the surface.

Subsequent to this discussion, we propose that there is a layer of thickness $(l = L\epsilon)$ near the surface of the sphere where the diffusion term is comparable to the convection term, and $\epsilon \ll 1$. Note that ϵ is a scale factor, which will be determined from the condition that the convective and diffusive terms are of equal magnitude in the thermal boundary layer. The scale factor ϵ is determined by a balance between convection and diffusion at the surface, and is dependent on the velocity boundary conditions at the surface. Two limiting cases can be considered,

1. Flow past a solid particle, in which case there is a no-slip condition at the surface, and the components of the fluid velocity normal and tangential to the surface are zero.

2. Flow past a gas bubble, in which case the component of the velocity normal to the surface is zero, but the component tangential to the surface is non-zero because the zero tangential stress boundary condition is applied on the liquid side of the gas-liquid interface.

1.4 Diffusion from a solid particle

Next, we analyse the streaming flow past a sphere in the high Peclet number limit. The configuration and coordinate system are shown in figure **??**, and all lengths are scaled by R, the radius of the sphere, and all velocities are scaled by U_∞, the free stream velocity at a large distance from the sphere. The components of the velocity for the creeping flow around the sphere are,

$$u_r^* = \left[1 - \frac{3}{2}\left(\frac{1}{r^*}\right) + \frac{1}{2}\left(\frac{1}{r^*}\right)^3\right]\cos(\theta) \tag{1.169}$$

$$u_\theta = -\left[1 - \frac{3}{4}\left(\frac{1}{r^*}\right) - \frac{1}{4}\left(\frac{1}{r^*}\right)^3\right]\sin(\theta) \tag{1.170}$$

It is clear, from the discussion in the previous section, that diffusion is comparable to convection only in a thin 'boundary layer' near the surface. Therefore, we define an 'inner coordinate' $y^* = (r-R)/R = ((1-r^*)/\epsilon)$, which represents the distance from the solid surface scaled by a small parameter ϵ. The value of ϵ will be determined by balancing convection and diffusion in the boundary layer. With this rescaling, we can express the equations for the velocity field in the inner region as,

$$u_r^* = \left(1 - \frac{3}{2(1+\epsilon y^*)} + \frac{1}{2(1+\epsilon y^*)^3}\right)\cos(\theta) \tag{1.171}$$

$$u_\theta^* = -\left(1 - \frac{3}{4(1+\epsilon y^*)} - \frac{1}{4(1+\epsilon y^*)^3}\right)\sin(\theta) \tag{1.172}$$

Since the parameter ϵ is small and we are interested in the region $y^* \sim 1$, the expressions for the components of the velocity, 1.171 and 1.172, can be simplified by retaining just the leading order terms in an expansion in ϵ,

$$u_r^* = \frac{3}{2}\epsilon^2 y^{*2}\cos(\theta) \tag{1.173}$$

$$u_\theta^* = -\frac{3}{2}\epsilon y^*\sin(\theta) \tag{1.174}$$

These expressions for the components of the velocity are inserted into the convection-diffusion equation, 1.145, and the radial coordinate is written in terms of the y^* using the transformation $r^* = (1+\epsilon y^*)$, to obtain,

$$\begin{aligned} \mathrm{Pe}\left(\frac{3}{2}\epsilon^2 y^{*2}\cos(\theta)\frac{1}{\epsilon}\frac{\partial\Theta}{\partial y^*} - \frac{3}{2}y^*\epsilon\sin(\theta)\frac{1}{(1+\epsilon y^*)}\frac{\partial\Theta}{\partial\theta}\right) \\ = \frac{1}{(1+\epsilon y^*)^2}\frac{1}{\epsilon}\frac{\partial}{\partial y^*}\left((1+\epsilon y^*)^2\frac{1}{\epsilon}\frac{\partial\Theta}{\partial y^*}\right) \\ +\frac{1}{(1+\epsilon y^*)^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Theta}{\partial\theta}\right) \end{aligned} \tag{1.175}$$

It is apparent that the term on the left side is proportional to Peϵ in the limit $\epsilon \to 0$, while the largest term on the right, which is the first term on the right, is proportional to $(1/\epsilon^2)$. A balance is achieved only for $\epsilon \sim \mathrm{Pe}^{-1/3}$, so that the thickness of the boundary layer decreases proportional to $R\mathrm{Pe}^{-1/3}$ in the limit Pe $\gg 1$. If we substitute $\epsilon = \mathrm{Pe}^{-1/3}$ in equation 1.175, and retain only the largest terms in the expansion, we get,

$$\frac{3}{2} y^{*2} \cos(\theta) \frac{\partial \Theta}{\partial y^*} - \frac{3}{2} y^* \sin(\theta) \frac{\partial \Theta}{\partial \theta} = \frac{\partial^2 \Theta}{y^{*2}} \tag{1.176}$$

This is the ‘boundary layer’ equation for the thermal boundary layer at the surface of the sphere.

In order to solve the equation, we note that there are boundary conditions on the surface of the sphere ($y^* = 0$) and in the limit $y^* \to \infty$. Since there is no length scale in the problem,we can examine a solution in terms of a similarity variable, η,

$$\Theta_0 = \Theta_0(\eta) \tag{1.177}$$

where the similarity variable is defined as,

$$\eta = (y^*/h(\cos(\theta)) \tag{1.178}$$

The derivatives of Θ_0 can then be expressed only in terms of the variable η,

$$\frac{\partial \Theta_0}{\partial y^*} = \frac{1}{h} \frac{\partial \Theta_0}{\partial \eta} \tag{1.179}$$

$$\frac{\partial^2 \Theta_0}{\partial y^{*2}} = \frac{1}{h^2} \frac{\partial^2 \Theta_0}{\partial \eta^2} \tag{1.180}$$

$$\frac{\partial \Theta_0}{\partial \theta} = -\frac{y^*}{h^2} \frac{dh}{d\theta} \frac{\partial \Theta_0}{\partial \eta} \tag{1.181}$$

Substituting this into the convection-diffusion equation 1.176, we get,

$$\frac{d^2 \Theta_0}{d\eta^2} - \frac{3}{2} \eta^2 \frac{d\Theta_0}{d\eta} \left(h^3 \cos(\theta) + \sin(\theta) h^2 \frac{dh}{d\theta} \right) = 0 \tag{1.182}$$

The above equation has a similarity solution only is the term in the braces, $(h^3 - \sin(\theta)^2 h^2 (dh/d\theta))$, is a constant. The numerical value of this constant can be chosen arbitrarily, and for convenience we choose,

$$\left(\sin(\theta)^2 h^2 \frac{dh}{d\theta} - h^3 \right) = 2 \tag{1.183}$$

This equation can be solved to obtain,

$$h(\theta) = \frac{1}{\sin(\theta)} \left(\frac{6\theta + 3\sin(2\theta)}{2} + C \right)^{1/3} \tag{1.184}$$

The constant C has to be determined from the boundary condition in the θ coordinate. This constant is determined from the condition at $\theta = 0$, which is the upstream stagnation point on the sphere. At this point, the derivative of h with respect to θ is well defined, along all approaches, only if $(dh/d\theta) = 0$. If we apply this condition, we find that the constant $C = 0$, so that the solution for $h(\theta)$ is,

$$h(\theta) = \frac{1}{\sin(\theta)} \left(\frac{6\theta + 3\sin(2\theta)}{2} \right)^{1/3} \tag{1.185}$$

With this substitution, the equation 1.182 reduces to,

$$\frac{d^2\Theta_0}{d\eta^2} + 3\eta^2 \frac{d\Theta_0}{d\eta} = 0 \tag{1.186}$$

This equation can be integrated once by parts to obtain,

$$\frac{d\Theta_0}{dt} = A \exp(-\eta^3) \tag{1.187}$$

After integrating a second time, we obtain the most general solution,

$$\Theta_0 = A \int_0^{\eta} d\eta' \exp(-\eta'^3) + B \tag{1.188}$$

The boundary conditions require that in the limit $y^* \to \infty$, $\Theta_0 = 0$, while $\Theta_0 = 1$ at $y^* = 0$. With these, the general solution for the temperature field becomes,

$$\Theta_0 = 1 - \frac{\int_0^{\eta} d\eta' \exp(-\eta'^3)}{\int_0^{\infty} d\eta' \exp(-\eta'^3)} \tag{1.189}$$

The radial heat flux at the surface is given by,

$$\begin{aligned} q_r &= -\frac{K}{R} \left. \frac{\partial T}{\partial r^*} \right|_{r^*=1} \\ &= -\frac{K(T_0 - T_\infty)}{R} \left. \frac{\partial \Theta}{\partial r^*} \right|_{r^*=1} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{K\mathrm{Pe}^{1/3}}{R}\left.\frac{d\Theta_0}{dy^*}\right|_{y^*=0} \\
&= -\frac{K\Pi^{1/3}}{Rh(\theta)}\left.\frac{dT_0}{d\eta}\right|_{\eta=0} \\
&= -\frac{KPe^{1/3}}{Rh(\theta)}\frac{1}{\int_0^\infty d\eta' \exp\left(-\eta'^3\right)} \qquad (1.190)
\end{aligned}
$$

This flux is a function of position θ on the surface, and so the total heat transferred from the surface to the fluid is given by,

$$Q = -\frac{KPe^{1/3}}{R}\frac{1}{\int_0^\infty d\eta' \exp\left(-\eta'^3\right)}R^2\int_0^{2\pi} d\phi \int_0^\pi \sin(\theta) d\theta \frac{1}{h(\theta)} \qquad (1.191)$$

It is necessary to evaluate the two integrals in the above expression numerically, and the numerical expressions for the integrals are,

$$\frac{1}{\int_0^\infty d\eta' \exp\left(-\eta'^3\right)} = \Gamma\left(\frac{4}{3}\right) = 0.8930 \qquad (1.192)$$

$$\int_0^\pi \sin(\theta) d\theta \frac{1}{h(\theta)} = 1.11546 \qquad (1.193)$$

Inserting the numerical values of the above integrals, we find the final expression for the total heat transferred from the particle is

$$Q = 1.2491(2\pi RK\mathrm{Pe}^{1/3})(T_0 - T_\infty) \qquad (1.194)$$

The Nusselt number is obtained by dividing this by

$$
\begin{aligned}
\mathrm{Nu} &= \frac{2Q}{(4\pi R^2 K(T_0 - T_\infty)/R)} \\
&= 1.2491\mathrm{Pe}^{1/3} \qquad (1.195)
\end{aligned}
$$

1.5 Diffusion from a gas bubble:

The Nusselt number for the diffusion from a gas bubble has a different dependence on the Peclet number than that from a solid particle, because the tangential velocity in the fluid at the bubble surface is not zero. Consequently, convective effects are stronger in the streaming past a gas bubble.

The coordinate system used is the same as that shown in figure **??**. The equations for the velocity field in the flow past a spherical bubble, with zero tangential stress conditions applied at the surface of the bubble, are different from those for the flow past a particle with no-slip conditions at the surface, 1.169 and 1.170.

$$u_r^* = \left(1 - \frac{1}{r^*}\right)\cos(\theta) \tag{1.196}$$

$$u_\theta = -\left(1 - \frac{1}{2r^*}\right)\sin(\theta) \tag{1.197}$$

As before, we focus attention on a thin region of thickness $R\epsilon$ near the surface of the bubble, where the small parameter ϵ will be determined by a balance between the convection and diffusion terms in the momentum conservation equation. The transformation to a scaled coordinate $y^* = (1 - r^*)/\epsilon$ is used, and the fluid velocity expressed in terms of these coordinates correct to leading order in an expansion in ϵ, are,

$$\begin{aligned} u_r^* &= \epsilon y^* \cos(\theta) \\ u_\theta^* &= -\frac{1}{2}(1 + \epsilon y^*)\sin(\theta) \end{aligned} \tag{1.198}$$

The expressions for the velocity are inserted into the convetion-diffusion equation, and the transformation $r^* = 1 + \epsilon y^*$ is used, to obtain,

$$\begin{aligned} &\mathrm{Pe}\left(\epsilon y^* \cos(\theta)\frac{1}{\epsilon}\frac{\partial \Theta}{\partial y^*} - \frac{1}{2}(1 + y^*\epsilon)\sin(\theta)\frac{1}{(1+\epsilon y^*)}\frac{\partial \Theta}{\partial \theta}\right) \\ &= \frac{1}{(1+\epsilon y^*)^2}\frac{1}{\epsilon}\frac{\partial}{\partial y^*}\left((1+\epsilon y^*)^2\frac{1}{\epsilon}\frac{\partial \Theta}{\partial y^*}\right) \\ &\quad + \frac{1}{(1+\epsilon y^*)^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial \Theta}{\partial \theta}\right) \end{aligned} \tag{1.199}$$

If we retain only the leading order terms in an expansion in small ϵ, we obtain,

$$\mathrm{Pe}\left(y^* \cos(\theta)\frac{\partial \Theta}{\partial y^*} + \frac{1}{2}\sin(\theta)\frac{\partial \Theta}{\partial \theta}\right) = \frac{1}{\epsilon^2}\frac{\partial^2 \Theta}{\partial y^{*2}} \tag{1.200}$$

It is clear that the leading order term on the left side is $O(\mathrm{Pe})$, while the leading order term in the right side is $O(1/\epsilon^2)$. Therefore, to achieve a balance between the convective and diffusive terms, it is necessary for $\epsilon \sim \mathrm{Pe}^{-1/2}$. This indicates that that the thickness of the thermal boundary later at the surface of a gas bubble is $O(\mathrm{Pe}^{-1/2})$, which is much small than the thickness $O(\mathrm{Pe}^{-1/3})$ at the surface of a particle. This is because the tangential velocity at the surface of a gas bubble is non-zero, in contrast to the zero tangential velocity at the surface of a solid particle. Consequently, convective effects are stronger at the surface of the gas bubble than at the surface of a solid particle.

We set $\epsilon = \mathrm{Pe}^{-1/2}$ in equation 1.200, to obtain,

$$\left(y^* \cos(\theta) \frac{\partial \Theta}{\partial y^*} + \frac{1}{2} \sin(\theta) \frac{\partial \Theta}{\partial \theta}\right) = \frac{\partial^2 \Theta}{\partial y^{*2}} \tag{1.201}$$

We explore the possibility of a similarity solution using the similarity variable $\eta = (y^*/h(\theta))$. The derivatives of Θ with respect to r^* and θ can be transformed using 1.179, 1.180 and 1.181, to obtain the convection-diffusion equation in terms of the similarity variable η,

$$\eta \frac{\partial \Theta}{\partial \eta} \left(h^2 \cos(\theta) + \frac{1}{2} h \frac{dh}{d\theta} \sin(\theta)\right) = \frac{\partial^2 \Theta}{\partial \eta^2} \tag{1.202}$$

This equation has a similarity solution only if $(h^2 \cos(\theta) + (1/2) h (dh/d\theta) \sin(\theta)$ is a constant. For convenience, we choose the constant to be -2,

$$\left(h^2 \cos(\theta) + \frac{1}{2} h \frac{dh}{d\theta} \sin(\theta)\right) = -2 \tag{1.203}$$

This equation has a solution of the form,

$$h(\theta) = \frac{1}{\sin(\theta)^2} (C + 9\cos(\theta) - \cos(3\theta))^{1/2} \tag{1.204}$$

The constant is determined from the boundary condition at the upstream stagnation streamline, $(dh/d\theta) = 0$ at $\theta = 0$. This requires that $C = -8$, and the final solution for $h(\theta)$ is,

$$h(\theta) = \frac{1}{\sin(\theta)^2} (9\cos(\theta) - \cos(3\theta) - 8)^{1/2} \tag{1.205}$$

With the substitution given in equation 1.203, equation 1.202 simplifies to,

$$\frac{\partial^2 \Theta_0}{\partial \eta^2} + 2\eta \frac{\partial \Theta}{\partial \eta} = 0 \tag{1.206}$$

with the boundary conditions $\Theta = 1$ at $\eta = 0$, and $\Theta \to 0$ as $\eta \to \infty$. This can be solved quite easily to obtain,

$$\Theta = 1 - \frac{\int_0^\eta d\eta' \exp{(-\eta'^2)}}{\int_0^\infty d\eta' \exp{(-\eta'^2)}} \tag{1.207}$$

1.6 Streaming past objects of arbitrary shape:

In this section, we examine the extent to which the previous results obtained for spherical particles and bubbles can be generalised to arbitrary shapes of particles, drops and bubbles.

First consider the case of the fluid flow past a solid particle in which the velocity field satisfies the no-slip condition at the surface. The characteristic length scale of the particle is L, and the fluid velocity at a large distance from the particle is U_∞ in the x direction, as shown in figure **??**. We know, from the previous discussions, that the concentration or temperature variations are restricted to a boundary layer of thickness ϵL at the surface, where ϵ is a small parameter to be determined from a balance between convection and diffusion. The variation of the fluid velocity field very near the surface can be inferred as follows. Consider a coordinate system at a point on the surface, where the coordinate normal to the surface is X and the coordinate tangential to the surface is Y, as shown in figure **??**. The components of the mean velocity are required, by the no-slip condition, to be equal to zero at the surface $Y = 0$. Therefore, very near the surface for $Y \ll L$, it is expected that the two components of the velocity are linear in the coordinate Y, i. e., $u_X = U_\infty A(X) Y$ and $u_Y = U_\infty B(X) Y$, where $A(X)$ and $B(X)$ are dimensionless functions of the tangential coordinate X. However, for an incompressible fluid, this velocity field does not satisfy the mass conservation condition for an incompressible fluid. The mass conservation condition, $\nabla.\mathbf{u} = 0$, written in terms of X and Y near the surface is,

$$\frac{\partial u_X}{\partial X} + \frac{\partial u_Y}{\partial Y} = 0 \tag{1.208}$$

If we assume $u_X = A(X)Y$, the above equation requires that $u_\zeta = (-(dA(X)/dX)Y^2/2)$. Therefore, the fluid velocity in a thin layer near the surface can be approximated as,

$$u_X \;\; = \;\; U_\infty A(X) X$$

$$u_Y = -U_\infty \frac{dA(X)}{dX} \frac{Y^2}{2} \tag{1.209}$$

The exact form of the function $A(X)$ depends on the details of the particle shape.

This velocity field is inserted into the convection-diffusion equation, to obtain,

$$U_\infty \left(A(\xi) Y \frac{\partial \Theta}{\partial X} - \frac{Y^2}{2} \frac{dA(X)}{dX} \frac{\partial \Theta}{\partial Y} \right) = \alpha \left(\frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Y^2} \right) \tag{1.210}$$

Here, we have neglected diffusion in the streamwise direction, since we know that this is small compared to diffusion in the cross-stream direction. It is convenient to define a scaled tangential coordinate $X^* = (X/L)$, since L is the length scale for the variation of temperature in the flow direction. In the cross-stream direction, we postulate that there is a much smaller length ϵL over which convection and diffusion are of equal magnitude in the high Peclet number limit, where $\epsilon \ll 1$. Consequently, the dimensionless distance in normal to the surface is defined as $\epsilon^* = (Y/\epsilon L)$. Inserting these into the convection diffusion equation, and multiplying throughout by $\epsilon^2 L^2$, we get,

$$\mathrm{Pe}\epsilon^3 \left(A(X^*) \epsilon^* \frac{\partial \Theta}{\partial X^*} - \frac{\epsilon^{*2}}{2} \frac{dA(X^*)}{dX^*} \frac{\partial \Theta}{\partial \epsilon^*} \right) = \left(\epsilon^2 \frac{\partial^2 \Theta}{\partial X^{*2}} + \frac{\partial^2 \Theta}{\partial \epsilon^{*2}} \right) \tag{1.211}$$

The above equation 1.211 indicates that there is a balance between convection and diffusion only for $\epsilon \sim \mathrm{Pe}^{-1/3}$, and $\epsilon^* = (\mathrm{Pe}^{1/3} Y/L)$. Thus, the boundary layer thickness is $O(\mathrm{Pe}^{-1/3})$ smaller than the particle length L for the general case of the flow past a solid surface. Without loss of generality, we set $\epsilon = \mathrm{Pe}^{-1/3}$, and neglect the first term on the right side of equation 1.211 since it is $O(\epsilon^2)$, to obtain the boundary layer equation,

$$\left(A(X^*) \epsilon^* \frac{\partial \Theta}{\partial X^*} - \frac{\epsilon^{*2}}{2} \frac{dA(X^*)}{dX^*} \frac{\partial \Theta}{\partial \epsilon^*} \right) = \frac{\partial^2 \Theta}{\partial \epsilon^{*2}} \tag{1.212}$$

To proceed further, we examine the possibility of a similarity solution, using the substitution,

$$\eta = \frac{\epsilon^*}{h(X^*)} \tag{1.213}$$

The derivatives of Θ with respect to X^* and ϵ^* can be expressed in terms of η using the chain rule,

$$\begin{aligned}\frac{\partial\Theta}{\partial\epsilon^*} &= \frac{1}{h}\frac{\partial\Theta}{\partial\eta} \\ \frac{\partial^2\Theta}{\partial\epsilon^{*2}} &= \frac{1}{h^2}\frac{\partial^2\Theta}{\partial\eta^2} \\ \frac{\partial\Theta}{\partial X^*} &= -\frac{\epsilon^*}{h^2}\frac{dh}{dX^*}\frac{\partial\Theta}{\partial\eta}\end{aligned} \tag{1.214}$$

This is inserted into equation 1.212, and the left and right sides are multiplied by h^2, to obtain,

$$\frac{\partial^2\Theta}{\partial\eta^2} + \eta^2\frac{\partial\Theta}{\partial\eta}\left(A(\xi)h(\xi)^2\frac{dh}{d\xi} + \frac{h(\xi)^3}{2}\frac{dA}{d\xi}\right) = 0 \tag{1.215}$$

The boundary conditions are,

$$\begin{aligned}\Theta &= 1 \text{ at} \epsilon^* = 0 \to \eta = 0 \\ \Theta &= 0 \text{ as} \epsilon^* \to \infty \to \eta \to \infty\end{aligned} \tag{1.216}$$

It is apparent that the equation 1.215 admits a similarity solution only if

$$\left(A(X^*)h(X^*)^2\frac{dh}{dX^*} + \frac{h(X^*)^3}{2}\frac{dA}{dX^*}\right) = \text{Constant} \tag{1.217}$$

The constant can be set to any value, without loss of generality, and it is convenient to set the constant equal to 3. With this, equation 1.217 is rewritten as,

$$\frac{dh(X^*)^3}{dX^*} + \frac{3h(X^*)^3}{2A(X^*)}\frac{dA}{dX^*} = \frac{9}{A(X^*)} \tag{1.218}$$

This equation can be solved to obtain,

$$h(X^*)^3 = 9A(X^*)^{-3/2}\int_0^{X^*} dX^{*\prime}A(X^{*\prime})^{1/2} + CA(X^*)^{-3/2} \tag{1.219}$$

The equation 1.215 for the temperature field becomes,

$$\frac{\partial^2\Theta}{\partial\eta^2} + 3\eta^2\frac{\partial T}{\partial\eta} = 0 \tag{1.220}$$

This can be easily solved to obtain,

$$\Theta = C_1 \int_0^{\eta} d\eta' \exp(-\eta'^3/3) + C_2 \tag{1.221}$$

The constants C_1 and C_2 are easily determined from the boundary conditions, 1.214, and the final expression for Θ is,

$$\Theta = 1 - \frac{\int_0^{\eta} d\eta' \exp(-\eta'^3/3)}{\int_0^{\eta} d\eta' \exp(-\eta'^3/3)} \tag{1.222}$$

The heat flux normal to the surface can now be calculated,

$$\begin{aligned} q_Y &= -K \left.\frac{dT}{dY}\right|_{Y=0} \\ &= -\frac{K(T_0 - T_\infty)}{L} \left.\frac{d\Theta}{d\epsilon^*}\right|_{\epsilon^*=0} \\ &= -\frac{K(T_0 - T_\infty)\mathrm{Pe}^{1/3}}{Lh(X^*)} \left.\frac{d\Theta}{d\eta}\right|_{\eta=0} \\ &= \frac{K(T_0 - T_\infty)\mathrm{Pe}^{1/3}}{Lh(X^*)} \frac{1}{\int_0^{\eta} d\eta' \exp(-\eta'^3/3)} \end{aligned} \tag{1.223}$$

The total heat generated by the object, per unit width in the direction perpendicular to the plane of flow, is,

$$Q = \frac{K(T_0 - T_\infty)\mathrm{Pe}^{1/3}}{L} \frac{1}{\int_0^{\eta} d\eta' \exp(-\eta'^3/3)} \int dX^* \frac{1}{h(X^*)} \tag{1.224}$$

Problems:

1. In a gas-liquid contacting system, a fluid film of thickness H is flowing down an inclined plane, and is in contact with the gas phase at the liquid-gas interface. The fluid velocity is given by:

$$u_x = \frac{3\bar{u}}{H^2}(yH - y^2/2)$$

where $\bar{u}$ is the mean velocity, x is the flow direction and y is perpendicular to the interface. At the liquid-gas interface, the concentration of the solute in the liquid phase is c_s. At the entrance, the concentration is zero throughout the film. The concentration field satisfies the equation,

$$\mathbf{u}.\nabla c = D\nabla^2 c$$

where $\mathbf{u}$ is the fluid velocity.

(a) Choose an appropriate coordinate system and write the steady state mass balance equation for this system.

(b) What is the dimensionless number that determines the ratio of convection and diffusion?

(c) If this number is large, one would expect the solute concentration to be confined to a thin layer in the liquid film near the liquid-gas interface. Scale the coordinates in the heat balance equation in this case, and determine the boundary layer thickness.

(d) If the concentration at a point on the plate is determined only by the conditions upstream of the point, and not by the total film thickness H, find a similarity equation for the concentration profile.

(e) Determine the mass flux for a film of length L, and the Nusselt number.

2. Consider the mass diffusion from a spherical bubble in the high Peclet number limit, governed by the convection-diffusion equation

$$\mathbf{u}.\nabla c = D\nabla^2 c$$

There is a constant velocity $u_z = U_\infty$ in the z direction, and the velocity field is given by,

$$u_r = U_\infty \left(1 - \frac{R}{r}\right) \cos(\theta)$$
$$u_\theta = -U_\infty \left(1 - \frac{R}{2r}\right) \sin(\theta)$$

where r is the radial position, R is the radius of the bubble, and θ is the angle from the z direction.

(a) In the high Peclet number limit, express $(r/R) = (1 + \epsilon y)$, where ϵ is a small coordinate. Insert this into the convection-diffusion equation, and obtain the dependence of ϵ on the Peclet number for a balance between convection and diffusion.

(b) Obtain the leading order equation in the high Peclet number limit. Use a similarity transform which relates the y and θ coordinates using $\eta = y/h(\theta)$. What equation has to be satisfied by $h(\theta)$ for a similarity solution to be possible?

(c) Solve the convection-diffusion equation in terms of the similarity variable.

3. Consider the flow around a cylinder as shown in figure **??**. The velocity field in the radial co-ordinate system is given by,

$$\begin{aligned} u_r &= -U\cos(\theta)\left(1 - \frac{R^2}{r^2}\right) \\ u_\theta &= U\sin(\theta)\left(1 + \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi r} \end{aligned} \tag{1.225}$$

where R is the radius of the cylinder. The cylinder surface is at a temperature T_0, while the temperature far from the cylinder is T_∞. The equation for the temperature field is the convection diffusion equation

$$\mathbf{u}.\nabla T = \alpha \nabla^2 T \tag{1.226}$$

(a) Insert the expression for the fluid velocity into the above equation, and scale the resulting equation to obtain a dimensionless equation for the temperature field. What is the Peclet number (ratio of convection and diffusion)?

(b) Consider the limit where the Peclet number is large. In this case, the temperature variation is expected to be confined to a thin boundary layer near the surface of the cylinder. Scale the distance from the surface of the cylinder by a boundary layer thickness, and simplify the equation. How is the boundary layer thickenss related to the Peclet number?

(c) Use a similarity transform to express the convection-diffusion equation in terms of the ratio of the distance from the surface and a boudary layer thickness, where the boundary layer thickness is a function of the θ co-ordinate. What is the equation for the variation for the boundary layer co-ordinate with θ? Solve this equation to determine the boundary layer thickness as a function of θ.

(d) Solve the equation for the temperature in terms of the similarity variable.

(e) Determine the Nusselt number as a function of the Peclet number for this case.