# Chapter 7

# Kinematics

## 7.1 Tensor fields

In fluid mechanics, the fluid flow is described in terms of 'vector fields' or 'tensor fields' such as velocity, stress, pressure, etc. It is important, at the outset, to discuss the terms 'tensor' and 'field' in some detail.

A 'field' is a just a smoothly varying function of space and time. For example, consider a jar of water heated from below with insulated sides as shown in figure 7.1. In the absence of convection, the temperature varies in a linearly from $T_b$ at the bottom to $T_t$ at the top. To describe this variation, one could use the coordinate system shown in figure 7.1, with vertical coordinate $x_3$ and horizontal coordinates $x_1$ and $x_2$. The temperature can then be represented by a smoothly varying function $T(x_1, x_2, x_3, t) = T_b - (T_b - T_a)(x_3/L)$. This smoothly varying function of position is the temperature 'field' in the jar. If there is convection, the temperature is no longer a function of $x_3$ alone, but could be a function of the coordinates $x_1, x_2$ and time. In fact, it may no longer be possible to represent the temperature field as a simple function of spatial and time coordinates. However, a differential equation for the temperature can still be formulated, and solved using a computer if necessary. Similarly, in case there is convection in the heated jar, the velocity can be described by the 'velocity field', which is a smoothly varying function of the spatial coordinates $(x_1, x_2, x_3)$ and time. Other variables of interest in fluid dynamics, such as the pressure and stress, can be formulated in a similar manner.

The description of fluid properties in terms of fields assumes that the fluid is a continuous medium in which the properties have definite values at every point in the fluids. This immediately raises the question, 'Under what conditions is this continuum approximation valid?' The pointwise description of flow variables involves a limiting process over the microscopic values of the flow variables about the point. For example, consider the value of the velocity $u_1$ in the $x_1$ direction at the point $(x_1, x_2, x_3)$. At a microscopic level, the fluid is not continuous, but is made up of discrete molecules, each of which has a velocity $u_1$. If we take a small volume $V$ about the point $(x_1, x_2, x_3)$ which contains $N$ molecules, the average velocity of the molecules can be defined as

$$\langle u_1 \rangle_N = \frac{1}{N} \sum_{\alpha=1}^{N} u_{\alpha 1} \tag{7.1}$$

where $u_{\alpha 1}$ is the velocity of the molecule with index $\alpha$, and $\langle u_1 \rangle_N$ denotes an average of the velocity $u_1$ over $N$ molecules. Obviously, the volume $V$ has to be small compared to the volume of the jar of water, because the average velocity is assigned to a point in the continuum approximation. If the volume $V$ is of molecular dimensions, the average will depend on the number of molecules in the volume. In the extreme case where the volume is of the same magnitude as a molecule, the average will be zero or non - zero depending on the presence or absence of a molecule in the volume. However, as the number of molecules in a volume becomes large, the fluctuations in the value of the average velocity become smaller, because the variation in the number of molecules is small compared to the total number of molecules $N$. In fact, there is a theorem called the 'Central Limit Theorem' which states that the standard deviation in the average velocity decreases proportional to $(1/\sqrt{N})$ as the number of molecules $N$ becomes large. Therefore, the continuum approximation is a good one when the two requirements, that (i) when the characteristic length of the averaging volume is small compared to the characteristic length scale for the flow (in this case the size of the jar) (ii) there are a large number of molecules in this volume, are simultaneously satisfied. This approximation is a good one in most practical applications. For example, the characteristic length of the flow is $1cm$, one could consider a cube of length $10^{-4}cm$ on each side as the averaging volume. In a volume of this size, there are of the order of $10^8$ molecules in a gas, and $10^{11}$ molecules in a liquid at room temperature and atmospheric pressure. From these figures, it is obvious that the continuum approximation is a good one for length scales corresponding to our everyday experience.

The continuum approximation is used throughout this course for analysing fluid flows. One advantage of this approach is that it is no longer necessary to worry about the microscopic details of molecular motion. The equations for the density, velocity and pressure fields are derived using mass, momentum and energy considerations and certain 'constitutive' relations which describe the deformation of a fluid element caused by imposed stresses. The governing equations for the fluid flow are in the form of partial differential equations(PDEs), and the techniques developed for the solution of PDEs can be used for determining the fluid velocity field. However, as mentioned before, physical insight into the nature of the flow is more important than mathematical manipulation, and the emphasis in this course will be on developing the physical intuition necessary for analysing the fluid flow.

### 7.1.1 Tensors

Quantities which are completely specified by their magnitude, such as mass and temperature, are called scalars. But the description of motion involves the concept of physical direction in addition to magnitude. Quantities which possess both magnitude and direction are called 'tensors' in general, and a special class of tensors are called 'vectors'. A brief review of vectors and tensors is given in this chapter, with the purpose of clarifying the basic concepts, and also to introduce some simplified notation which makes vector algebra a lot easier. Vectors are easier to visualise, and so we will begin with a discussion of vectors and then generalise the concepts to tensors. Since only three dimensional geometries are considered in this course, the discussion is restricted to vectors and tensors in three dimensions. In addition, we consider only orthogonal coordinate systems, where the unit vectors are perpendicular to each other. The understanding of non - orthogonal coordinate systems requires some more advanced concepts which will not be discussed here.

Vectors and tensors are usually expressed in terms of their components in the coordinate directions in a coordinate system. However, it is important to note that vectors and tensors *possess certain intrinsic properties that are independent of the coordinate system in which it is described.* If the vector is visualised as a directed line at a point, then the length of the line (which represents the 'magnitude' of the vector) is invariant under translation and rotation of the coordinate system. In addition, the angle between two vectors, which is related to the 'dot product' (which will be discussed a little later), is also invariant under arbitrary translation and rotation of the coor-

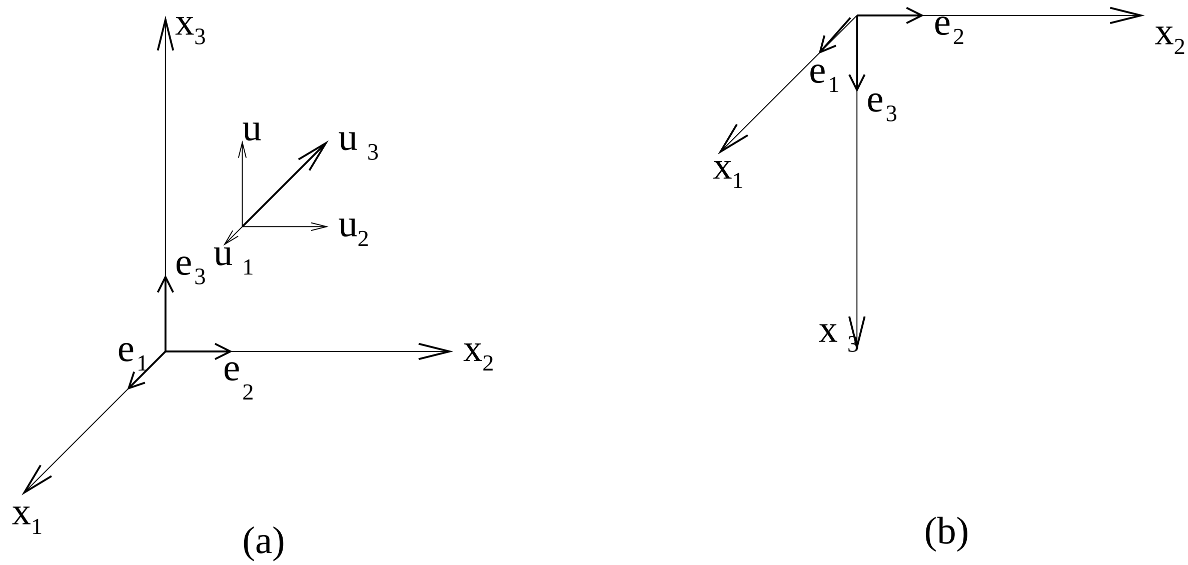


Figure 7.1: Coordinate axes and unit vectors in a right handed (a) and left handed (b) coordinate system.

dinate system. Thus, a vector has certain properties that are independent of the coordinate system, and the laws of fluid mechanics which relate different vectors or tensors are independent of the coordinate system used for the analysis. For ease of visualisation, a Cartesian coordinate system is first used for a discussion of tensors The Cartesian coordinate system shown in figure 7.1(a), with coordinate axes are $x_1$, $x_2$ and $x_3$, is used in this course. Note that this is a 'right handed' coordinate system, as opposed to figure 7.1(b) which is 'left handed' coordinate system. The unit vectors, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, have unit length and are directed along the $x_1$, $x_2$ and $x_3$ directions respectively.

A vector is a quantity that is represented by a line that has a certain magnitude, and has one physical direction associated with it. For example, the velocity vector of a particle has a magnitude, which is the speed, and a physical direction, which is the direction in which the particle moves. It is common to describe a vector in three dimensional space as the sum of three components. For example, in Cartesian coordinate system shown in figure 7.1(a), a velocity vector $\mathbf{u}$ can be written as:

$$\begin{aligned} \mathbf{u} &= u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 \\ & \sum_{i=1}^{3} u_i\mathbf{e}_i \end{aligned} \tag{7.2}$$

where $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ are the mutually perpendicular 'unit vectors' in three dimensions.

A tensor is the generalisation of a vector, which could have one or more 'physical directions' associated with it. The 'order' of the tensor is the number of physical directions associated with the tensor. A vector is a first order tensor, which has one direction associated with it. An example of a second order tensor is the 'stress tensor' $\sigma$, which is the force per unit area in a fluid acting at a surface. The force acting at a surface due to fluid flow depends on the flow conditions and the orientation of the surface, and there are two directions associated with the stress, (i) the direction of the force and (ii) the direction of the unit normal to the surface which gives the orientation of the surface. The stress tensor can be written, in a manner similar to 7.2, as

$$\begin{aligned} \sigma &= \sigma_{11}\mathbf{e}_1\mathbf{e}_1 + \sigma_{12}\mathbf{e}_1\mathbf{e}_2 + \sigma_{13}\mathbf{e}_1\mathbf{e}_3 + \sigma_{21}\mathbf{e}_2\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2\mathbf{e}_2 + \sigma_{23}\mathbf{e}_2\mathbf{e}_3 + \sigma_{31}\mathbf{e}_3\mathbf{e}_1 + \sigma_{32}\mathbf{e}_3\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3\mathbf{e}_3 \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij}\mathbf{e}_i\mathbf{e}_j \end{aligned} \tag{7.3}$$

where $\sigma_{ij}$ is the force per unit area in the $i$ direction acting at a surface whose unit normal is in the $j$ direction.

A tensor can also be formed by a 'tensor product' of two or more vectors or tensors.

$$\mathbf{ab} = \sum_i \sum_j a_i b_j \mathbf{e}_i\mathbf{e}_j \tag{7.4}$$

**Note that the tensor product of two vectors is a second order tensor, which is very different from the dot product which is a scalar.** A second order tensor is also obtained when the gradient operator acts on a vector. For example, the gradient of the velocity,

$$\nabla\mathbf{v} = \sum_i \sum_j \mathbf{e}_i\mathbf{e}_j \frac{\partial u_i}{\partial x_j} \tag{7.5}$$

is a second order tensor. In this, the index $i$ represents the direction of the velocity, while the direction $j$ represents the direction in which the partial derivative is evaluated. The nine components of this tensor contain information about the variation of the velocity about a point. For example, the variation of the velocity about the point $(x_1, x_2, x_3)$ can be written as

$$\begin{aligned}
\Delta\mathbf{v} &= \sum_i \mathbf{e}_i \frac{\partial v_i}{\partial x_j}\Delta x_j \\
&= (\sum_i \sum_j \mathbf{e}_i\mathbf{e}_j \frac{\partial u_i}{\partial x_j}).(\sum_j \mathbf{e}_j \Delta x_j) \\
&= (\nabla\mathbf{v}).\Delta\mathbf{x} \qquad (7.6)
\end{aligned}$$

This second order tensor contains information about the variation of the velocity vector about a point. For example, the rate of change of velocity with distance between the point $(x_1, x_2, x_3)$ and $(x_1 + \Delta x_1, x_2, x_3)$ is given by

$$\frac{\mathbf{v}(x_1 + \Delta x_1, x_2, x_3) - \mathbf{v}(x_1, x_2, x_3)}{\Delta x_1} = \left(\mathbf{e}_1 \frac{\partial v_1}{\partial x_1} + \mathbf{e}_2 \frac{\partial v_1}{\partial x_2} + \mathbf{e}_3 \frac{\partial v_1}{\partial x_3}\right)\Delta x_1 \qquad (7.7)$$

The terms on the right side of equation 7.7 are, respectively, the $(1,1), (1,2), (1,3)$ components of the rate of deformation tensor 7.5. Similarly, the rate of change of velocity with distance in the $x_2$ direction is related to the $(2,1), (2,2), (2,3)$ components of the velocity gradient tensor, and the rate of change of velocity in the $x_3$ direction is related to the $(3,1), (3,2), (3,3)$ components of the velocity gradient tensor. Thus, the velocity gradient tensor contains information of the variation of all components of the velocity in all coordinate directions, as shown in figure 7.2. These could, of course, be expressed as a combination of partial derivatives of the different components in the different directions. However, there are significant advantages to expressing this as a tensor, because a tensor has certain measures which are independent of coordinate system. For example, the divergence of the velocity, which is the sum of the diagonal elements of the velocity gradient tensor, is independent of the choice of coordinate system. We will carry out a more detailed analysis of the velocity gradient tensor a little later.

The 'dot product' of two vectors $\mathbf{a}.\mathbf{b}$ is given by $|\mathbf{a}||\mathbf{b}|\cos(\theta)$ where $|\mathbf{a}|$ and $|\mathbf{b}|$ are the magnitudes of the two vectors, and $\theta$ is the angle between them. The orthogonality of the basis vectors can be easily represented by the following equation:

$$\mathbf{e}_i.\mathbf{e}_j = \delta_{ij} \qquad (7.8)$$

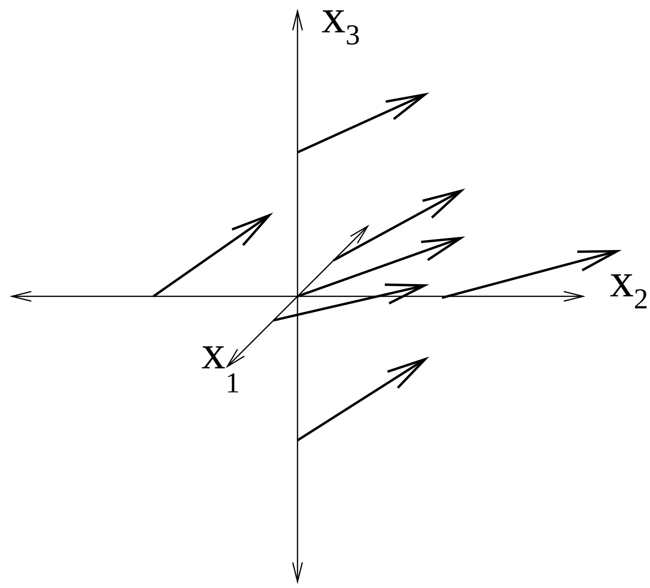


Figure 7.2: Variation of the velocity vector in different directions at a point.

where $\delta_{ij}$ is the 'Kronecker delta' which is given by:

$$\begin{aligned} \delta_{ij} &= 1 \quad \text{for} i = j \\ &= 0 \quad \text{for} i \neq j \end{aligned} \tag{7.9}$$

The orthogonality condition simply expresses the fact that the dot product of a unit vector with itself is 1, while the dot product of two different unit vectors is zero since they are orthogonal.

The dot product of two vectors **a** and **b** can now be expressed in using the Kronecker delta as follows:

$$\begin{aligned} \mathbf{a}.\mathbf{b} &= \left(\sum_i a_i \mathbf{e}_i\right)\left(\sum_j b_j \mathbf{e}_j\right) \\ &= \sum_i \sum_j a_i b_j \delta_{ij} \\ &= \sum_i a_i b_i \end{aligned} \tag{7.10}$$

Note:It is important to use different indices (in this case i and j) for the two summations. Use of the same index will lead to wrong (and often humerous) results. Another vector product which is used often is the 'cross product'. It is conventionally written in the following matrix form:

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_e \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \tag{7.11}$$

An alternative and easier way of writing the cross product uses the summation notation that has just been derived:

$$\mathbf{a} \times \mathbf{b} = \sum_i \sum_j \sum_k \epsilon_{ijk} a_j b_k \tag{7.12}$$

where the 'antisymmetric tensor' is defined as

$$\begin{aligned} \epsilon_{ijk} &= 1 \text{ for } (ijk) = (123), (231), (312) \\ &= -1 \text{ for } (ijk) = (132), (213), (321) \\ &= 0 \text{ otherwise} \end{aligned} \tag{7.13}$$

Note that the value of $\epsilon_{ijk}$ is equal to zero if any two of the indices, $(i, j, k)$, are repeated. This third order tensor is called the 'antisymmetric tensor' because the value of $\epsilon_{ijk}$ changes sign if any two of the indices are interchanged.

At this point, we introduce the 'indicial notation' which greatly simplifies the manipulation of vector equations. The convention is as follows.

1. The set of components of a vector $(a_1, a_2, a_3)$ is simply represented by $a_i$. In other words, a variable $a_i$ with a free variable represents the set $(a_1, a_2, a_3)$ and is equivalent to $\mathbf{a}$.

2. A subscript that is repeated twice implies a summation. For example,

$$aib_i \equiv a_1b_1 + a_2b_2 + a_3b_3 \tag{7.14}$$

With this convention, the dot product of two vectors can simply be written as:

$$\mathbf{a}.\mathbf{b} \equiv a_i b_i \tag{7.15}$$

without any necessity for cumbersome summations. The Einstein notation can also be used for tensors. For example, the second order tensor product of two vectors can be represented simply as:

$$\mathbf{ab} \equiv a_i b_j \tag{7.16}$$

In general, a tensor of order n will have n unrepeated indices, $i, j, \ldots$. When using the Einstein notation for large expressions, it is necessary to keep in mind two rules:

1. If an index appears just once in the expression, it represents three vector components, and if it appears twice, it represents a dot product of two vector components which is a scalar. If it appears more than twice, it means that you have made a mistake.

2. The indices that appear just once have to be identical on both sides of the equation. This is because a vector can be equated only to a vector, and a tensor of order n can be equated only to another tensor of order n. It does not make sense to equate a vector to a scalar or to a tensor of different order.

For example, the following are valid equations:

$$\begin{aligned} T_{ij} &= a_i b_j \\ L_{mn} &= K_{mp} J_{pn} \\ A_{ijkl} &= a_i b_{lm} c_{kl} d_m \end{aligned} \tag{7.17}$$

while the following are not correct equations:

$$\begin{aligned} T_{ij} &= a_i b_j c_l \\ B_{mnp} &= a_m b_{nl} c_{pl} d_l \end{aligned} \tag{7.18}$$

The cross product of two vectors, $\mathbf{a} \times \mathbf{b}$, can also be expressed quite easily using the indicial notation:

$$\mathbf{a} \times \mathbf{b} = \epsilon_{ijk} a_j b_k \tag{7.19}$$

The components of a vector (or tensor) depend, in general, on the coordinate system that we have chosen. Using the Einstein notation, it is easy to convert from one coordinate system to another. For example, consider a vector $\mathbf{a}$ has components $a_i$ in a coordinate system with unit vectors $\mathbf{e}_i$. If we are interested in the components $a'_i$ in a coordinate system with unit vectors $\mathbf{e}'_i$, the relation between the components in the two coordinate systems is simply given by:

$$a'_i = \beta_{ij} a_j \tag{7.20}$$

where the 'direction cosines' $\beta_{ij}$ are given by:

$$\beta_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j \tag{7.21}$$

The same procedure can be used for tensors as well.

The magnitudes of the components of vectors and tensors depend, in general, on the coordinate system being used for their description. However, there are certain tensors, called 'isotropic' tensors, components are invariant under changes in the coordinate system. These are derived from the symmetric tensor $\delta_{ij}$ or the antisymmetric tensor $\epsilon_{ijk}$. For example, $\lambda\delta_{ij}$, $\lambda(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$ and $\lambda\epsilon_{ijk}$ are all isotropic tensors. There is one property of antisymmetric tensors which proves useful in tensor calculations. Since the value of $\epsilon_{ijk}$ depends on the permutations of the indices $\{ijk\}$, the sign of the antisymmetric tensor changes if the coordinate system is changed from a right handed to a left handed coordinate system. However, a real tensor should be invariant such a transformation, and therefore the antisymmetric tensor all vectors derived from this are called 'pseudo vectors'. For example, if $\mathbf{u}$ and $\mathbf{v}$ are two real vectors, then the cross product of these $\mathbf{w_i} = \epsilon_{ijk}u_j v_k$ is a pseudo vector, i. e. $\mathbf{w}$ changes sign upon transformation from a right to a left handed coordinate system. A product containing an odd number of pseudo vectors is a pesudo vector, while a product containing an odd number of pseudo vectors is a real vector. Note that one cannot equate a real vector to a pseudo vector, because the equality becomes invalid upon transformation from a real to an imaginary coordinate system.

### 7.1.2 Vector calculus

There are three important derivatives that are encountered in fluid mechanics. The gradient of a scalar, $\phi$, is defined as:

$$\nabla\phi = \frac{\partial\phi}{\partial x_i} \equiv \partial_i\phi \tag{7.22}$$

Note that $\nabla\phi$ is a vector. The variation in $\phi$ due to a variation in the position $dx_i$ can be written as:

$$d\phi = (\partial_i\phi)dx_i \tag{7.23}$$

Since $d\phi$ is the dot product of $\partial_i\phi$ with $dx_i$, the vector $\nabla\phi$ is perpendicular to surfaces of constant $\phi$.

The divergence of a vector $a_i$ is defined as:

$$\nabla bf \cdot a = \partial_i a_i \tag{7.24}$$

The curl of a vector, $\nabla \times \mathbf{a}$, is defined as:

$$\nabla \times \mathbf{a} = \epsilon_{ijk}\partial_j a_k \tag{7.25}$$

Finally, the Laplacian of a scalar, $\nabla^2\phi$, is given by:

$$\nabla^2\phi = \partial_i^2\phi \tag{7.26}$$

In the above expressions, the scalar and vectors $\phi$ and $a_i$ can, in general be replaced by higher order tensors, as we shall see shortly.

We now move on to the 'integral theorems', which are useful results for the vector derivatives. The 'divergence theorem', also called the Gauss' or Green's theorem, is an extension of the elementary result:

$$\int_a^b \left(\frac{df}{dx}\right) dx = f(a) - f(b) \tag{7.27}$$

Consider a volume V enclosed by a surface S, and let $n_i$ be the outward normal to the surface. Consider a vector $a_i$ whose components and their first derivatives are continuous everywhere in the domain. Then, the theorem states that:

$$\int_V dV \partial_i a_i = \int_S dS n_i a_i \tag{7.28}$$

This gives a relation between the volume integral of the divergence of the vector and the surface integral of the vector itself. It is rather surprising that the volume integral of the divergence of $a_i$ depends only on the surface integral of $a_i$. This theorem can be easily extended to higher order tensors. If $a_{jklm}$ is an arbitrary tensor whose components and first derivatives are contiuous, then the divergence theorem for this tensor can be written as:

$$\int_V dV\ \partial_i a_{jklm} = \int_S dS\ n_i a_{jklm} \tag{7.29}$$

The Stokes theorem relates the curl of a vector on a surface to its value along the perimeter of the surface. Let C be any closed curve and S a surface that is bounded by the closed curve. Let $a_i$ be a vector whose components and first derivatives are continuous along S. Then the Stokes theorem states that:

$$\int_C dx_i\, a_i = \epsilon_{ijk} \int_S dS n_i\, \partial_j a_k \tag{7.30}$$

The convention usually used for the above integral is that the direction of the normal $n_i$ is dictated using the right hand rule from a knowledge of the directon of integration along the contour $C$.

## 7.2 Kinematics

This is the subject of the description of motion, without reference to the forces that cause this motion. Here, we assume that time and space are continuous, and identify a set of particles by specifying their location at a time $\{x_{i\alpha}^{(0)}, t_0\}$ for $\alpha = 1, 2, \ldots$. As the particles move, we follow their positions as a function of time $\{x_{ii}, t\}$. These positions are determined by solving the equations of motion for the particles, which we have not yet derived. Thus, the positions of the particles at time $t$ can be written as:

$$x_{i\alpha} = x_{i\alpha}(x_{i\alpha}^{(0)}, t) \tag{7.31}$$

Note that $x_{ii}$ and $t$ are independent variables; we can find the location of the particle only if the initial location $x_{ii}^{(0)}$ is given. Further, we can also express the (initial) position of the particles at time $t_0$ as a function of their (final) positions at $t$:

$$x_{i\alpha}^{(0)}(t_0) = x_{i\alpha}^{(0)}(x_{i\alpha}, t_0) \tag{7.32}$$

### 7.2.1 Lagrangian and Eulerian descriptions

The properties of the fluid, such as the velocity, temperature, etc. can be expressed in two ways. One is called the 'Lagrangian description', which is a natural extension of solid mechanics, where attention is focussed on a set of particles in the flow with initial position $x_i^{(0)}$, and the evolution of the properties of these particles as they move through space is determined, as shown in figure 7.3. For example, the temperature in Lagrangian variables is given by:

$$T = T(x_{ii}^{(0)}, t) \tag{7.33}$$

The positions of the particles can be expressed in the Lagrangian variables as noted above.

$$x_{ii} = x_{ii}(x_{ii}^{(0)}, t) \tag{7.34}$$

The velocity and acceleration are just time derivatives of the position of the particles:

$$v_{ii} = \partial_t x_{ii} a_{ii} = \partial_t^2 x_{ii} \tag{7.35}$$

In the 'Eulerian description', the positions of the properties of the fluid are expressed with reference to positions fixed in space. For example, the

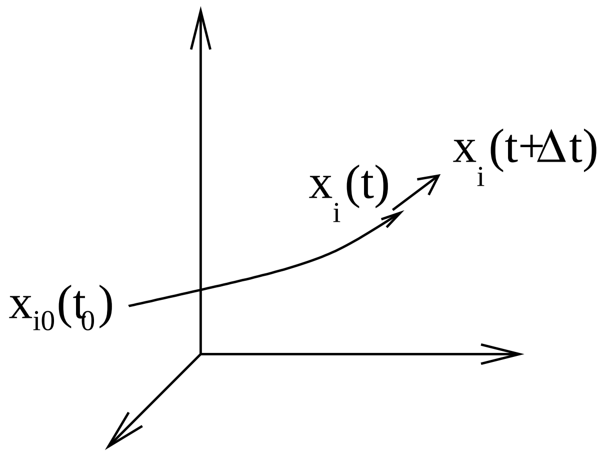


Figure 7.3: Eulerian and Lagrangian descriptions of variables in a fluid.

velocity and temperature fields are:

$$v_{ii} = v_{ii}(x_{ii}, t)T = T(x_{ii}, t) \tag{7.36}$$

The position vector in Eulerian variables is simply the particle position.

In order to illustrate the Lagrangian viewpoint, let us take a simple example. Consider a simple linear flow between two flat plates of length $L$ separated by a distance $H$, with the upper plate moving at a constant velocity $V$. Neglecting entrance effects, the fluid velocity is given by:

$$v_x = V(z/H)v_y = 0v_z = 0 \tag{7.37}$$

In addition, the fluid is being heated on the right and cooled on the left, so that there is a constant temperature gradient

$$T = T_0 + T_1 x \tag{7.38}$$

The velocity and temperature as specified above are Eulerian, because, they are referenced to a fixed coordinate.

To obtain the Lagrangian description, we consider a fluid particle with initial position $x_0, y_0, z_0$. The velocity of this particle is given by:

$$v_x = V(z_0/H)v_y = 0v_z = 0 \tag{7.39}$$

The velocity of the fluid particle remains constant as the particle moves through the channel, and is independent of time. The particle position is a function of time, however, and is given by:

$$x = x_0 + tV(z_0/H)y = y_0 z = z_0 \tag{7.40}$$

From the equation for the temperature profile along the length of the channel, we can obtain the Lagrangian form of the temperature profile as well:

$$T = T_0 + T_1 x = T_0 + T_1[x_0 + tV(z_0/H)] \tag{7.41}$$

This gives the Lagrangian form of the particle position, velocity and temperature for the simple flow that we have considered. For more complex flows, it is very difficult to obtain the Lagrangian description of the particle motion, and this description is not often used.

### 7.2.2 Substantial derivatives

Next, we come to the subject of the time derivatives of the properties of a fluid flow. In the Eulerian description, we focus only on the properties as a function of the positions in space. However, note that the position of the fluid particles are themselves a function of time, and it is often necessary to determine the rate of the change of the properties of a given particle as a function of time, as shown in figure 7.3. This derivative is referred to as the Lagrangian derivative or the substabtial derivative:

$$D_t A = \{\partial_t A\}|_{x_{ii}^{(0)}} \tag{7.42}$$

where $A$ is any general property, and we have explicitly written the subscript to note that the derivative is taken in the Lagrangian viewpoint, following the particle positions in space. For example, consider the substabtial derivative of the temperature of a particle as it moves along the flow over a time interval $\Delta t$:

$$\begin{aligned} D_t T &= \lim_{\Delta t \to 0} \left[ \frac{T(x_{ii} + \Delta x_{ii}, t + \Delta t) - T(x_{ii}, t)}{\Delta t} \right] \\ &= \lim_{\Delta t \to 0} \left[ \frac{T(x_{ii} + \Delta x_{ii}, t + \Delta t) - T(x_{ii}, t + \Delta t) + T(x_{ii}, t + \Delta t) - T(x_{ii} t)}{\Delta t} \right] \\ &= \partial_t T + \lim_{\Delta t \to 0} \frac{\Delta x_{ii}}{\Delta t} = \partial_t T + v_i \partial_i T \end{aligned} \tag{7.43}$$

Similarly, the acceleration of a particle in the Lagrangian viewpoint is the substantial derivative of the velocity:

$$a_{ii} = D_t v_{ii} = \partial_t v_{ii} + v_{ji} \partial_j v_{ii} \tag{7.44}$$

The relation between the Eulerian and Lagrangian derivative for the problem just considered can be easily derived. In the present example, the Eulerian derivative of the temperature field is zero, because the temperature field has attained steady state. The substabtial derivative is given by:

$$D_t T = v_i \partial_i T = v_x \partial_x T = (Vz/H) T_1 \tag{7.45}$$

This can also be obtained by directly taking the time derivative of the temperature field in the Lagrangian description.

### 7.2.3 Decomposition of the strain rate tensor

The 'strain rate' tensor refers to the relative motion of the fluid particles in the flow. For example, consider a differential volume $dV$, and a two particles located at $x_i$ and $x_i + dx_i$ in the volume, separated by a short distance $dx_i$. The velocity of the two particles are $v_i$ and $v_i + dv_i$. The relative velocity of the particles can be expressed in tensor calculus as:

$$dv_j = (\partial_i v_j) dx_i \tag{7.46}$$

In matrix notation, this can be expressed as:

$$\begin{pmatrix} dv_1 \\ dv_2 \\ dv_3 \end{pmatrix} = \begin{pmatrix} \partial_1 v_1 & \partial_2 v_1 & \partial_3 v_1 \\ \partial_1 v_2 & \partial_2 v_2 & \partial_3 v_2 \\ \partial_1 v_3 & \partial_2 v_3 & \partial_3 v_3 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} \tag{7.47}$$

The second order tensor, $(\partial_i v_j)$, can be separated into two components, a symmetric and an antisymmetric component, $\partial_i v_j = S_{ij} + A_{ij}$, which are given by:

$$S_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i) \tag{7.48}$$

$$A_{ij} = \frac{1}{2}(\partial_i v_j - \partial_j v_i) \tag{7.49}$$

The antisymmetric part of the strain rate tensor represents rotational flow. Consider a two dimensional flow field in which the rate of deformation tensor is antisymmetric,

$$\begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} \tag{7.50}$$

The flow relative to the origin due to this rate of deformation tensor is shown in figure 7.4(a). It is clearly seen that the resulting flow is rotational, and the angular velocity at a displacement $\Delta r$ from the origin is $a\Delta r$ in the anticlockwise direction. It can be shown that the antisymmetric part of the rate of deformation tensor is related to the vorticity, $\omega_i$, which is the curl of the velocity,

$$\begin{aligned} \omega_i &= \epsilon_{ijk}\frac{\partial u_k}{\partial x_j} \\ &= \frac{1}{2}\left(\epsilon_{ijk}\frac{\partial u_k}{\partial x_j} + \epsilon_{ikj}\frac{\partial u_j}{\partial x_k}\right) \\ &= \frac{1}{2}\left(\epsilon_{ijk}\frac{\partial u_k}{\partial x_j} - \epsilon_{ijk}\frac{\partial u_j}{\partial x_k}\right) \\ &= \frac{1}{2}\epsilon_{ijk}A_{kj} \end{aligned} \tag{7.51}$$

The relative velocity due to the antisymmetric part of the strain tensor is given by:

$$\Delta v_i = a_{ij}\Delta r_j = \frac{1}{2}\epsilon_{ijl}\omega_l \Delta r_j \tag{7.52}$$

or in more familiar vector notation,

$$\Delta \mathbf{v} = \frac{1}{2}\Delta \mathbf{r} \times \omega \tag{7.53}$$

Thus, the angular velocity is equal to half the vorticity.

The symmetric part of the stress tensor can be further separated into two components as follows:

$$S_{ij} = E_{ij} + \frac{1/3}{\delta}_{ij} S_{kk} \tag{7.54}$$

where $S_{ij}$ is the symmetric traceless part of the strain tensor, and $(1/3)\delta_{ij}S_{kk}$ is the isotropic part. 'Traceless' implies that the trace of the tensor, which

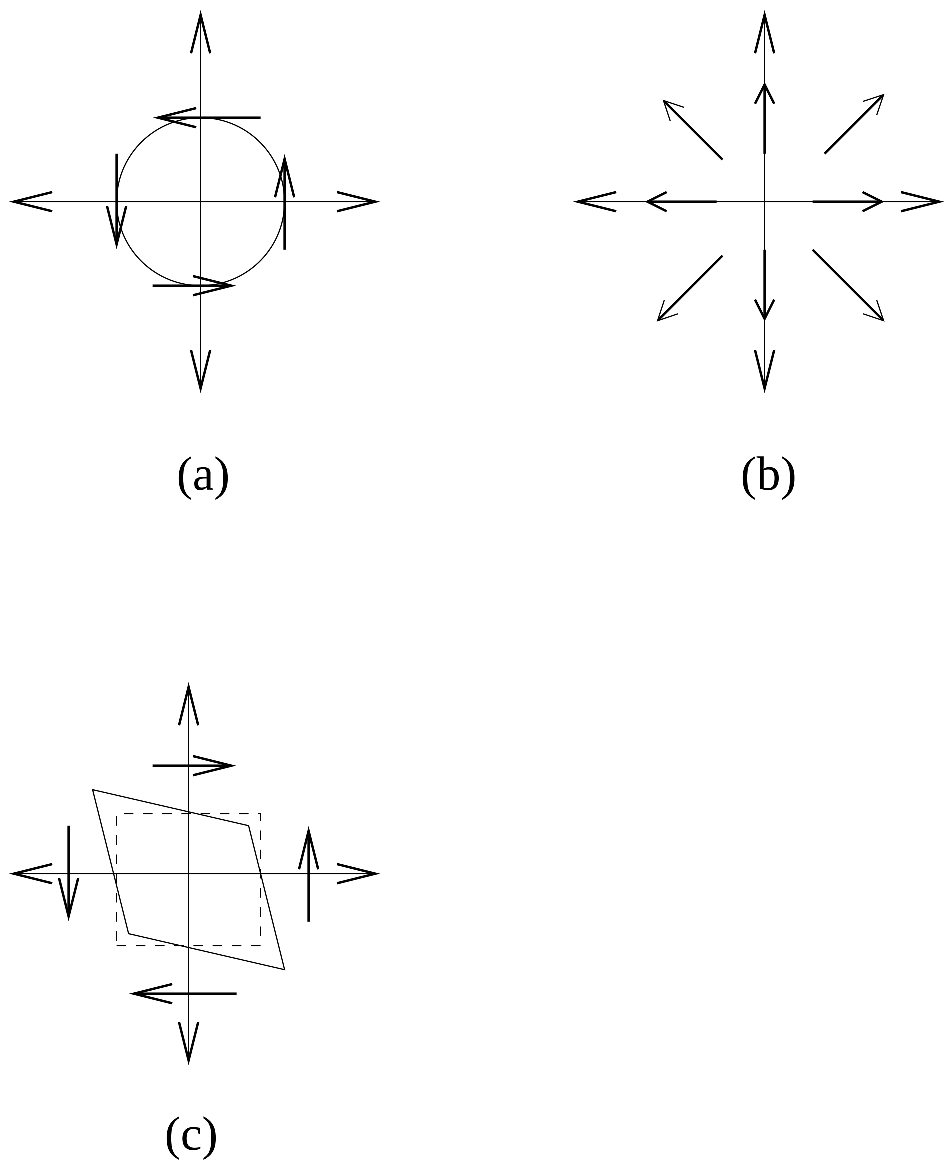


Figure 7.4: Velocity fields due to the antisymmetric, isotropic and the symmetric traceless parts of the rate of deformation tensor.

is $\delta_{ij}E_{ij} = E_{ii}$, is zero. To verify this, take the double dot product of the $S_{ij}$ with the identity tensor:

$$\begin{aligned} \delta_{ij}S_{ij} &= \delta_{ij}E_{ij} + \tfrac{1}{3}\delta_{ij}\delta_{ij}S_{kk} \\ S_{ii} &= E_{ii} + \tfrac{1}{3}\delta_{ii}S_{kk} \end{aligned} \tag{7.55}$$

The first component on the right side of the above equation, which is the symmetric traceless part of the rate of deformation tensor $E_{ij}$, is called the 'extensional strain', while the isotropic part, $(\delta_{ij}S_{kk}/3)$, corresponds to radial motion.

The velocity difference between the two neighbouring points due to the isotropic part of the rate of deformation tensor is given by

$$\begin{aligned} \Delta v_i &= \tfrac{1}{3}\delta_{ij}S_{kk}\Delta r_j \\ &= \tfrac{1}{3}S_{kk}\Delta r_i \end{aligned} \tag{7.56}$$

The above equation implies that the relative velocity between two points due to the isotropic component is directed along their line of separation, and this represents a radial motion, as shown in figure 7.4(b). The radial motion is outward if $S_{kk}$ is positive, and inward if $S_{kk}$ is negative. It can also be inferred, from the mass conservation equation, that the isotropic part is related to variations in density within the fluid. The mass conservation equation states that

$$\frac{D\rho}{Dt} + \rho\nabla.\mathbf{u} = 0 \tag{7.57}$$

which can also be written as

$$\frac{D\rho}{Dt} + \rho S_{kk} = 0 \tag{7.58}$$

since $S_{kk} = (\partial u_k/\partial x_k)$. The first term on the left of the above equation represents the rate of change of density in a reference frame moving with the fluid element. If there is radially outward motion from a fluid element, then $S_{kk}$ is positive and there is a decrease in the density within that element. Conversely, if there is radially inward motion, $S_{kk}$ is negative and the density increases within this moving element.

The symmetric traceless part represents an 'extensional strain', in which there is no change in density and no solid body rotation. In two dimensions, the simplest example of the velocity field due to a symmetric traceless rate of deformation tensor is

$$\begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} = \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} \tag{7.59}$$

The relative velocity of points near the origin due to this rate of deformation tensor is shown in figure 7.4(c). It is found that the fluid element near the origin deforms in such a way that there is no rotation of the principle axes, and there is no change in the total volume. This type of deformation is called 'pure extensional strain' and is responsible for the internal stresses in the fluid.

**Problems:**

1. 
   - Verify if the following expressions for tensors are correct, and determine their order.

     (a) $A_{ijkl}B_{mk}$ (b) $L_{ijm}K_{imn}M_{kmn}$

     (c) $S_{ijil}H_{jml}$ (d) $X_{ij}Y_{il}Z_{jl}$

   - Does the order of appearance of the components make a difference in the following expressions

     (a) $\epsilon_{ijk}a_j b_k$ and $b_k a_j \epsilon_{ijk}$

     (b) $\rho\partial_i v_j$ and $v_j\partial_j\rho$

     (c) $a_{ij}b_k$ and $a_{ik}b_j$

2. The stresses acting on the faces of a cube of unit length in all three directions are as follows:

$$T_{ij} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix}, \tag{7.60}$$

   where $i$ denotes the direction of the force and $j$ denotes the direction of normal to the area. Find out the force acting on the faces of the cube, and the torque on the cube in the three directions. For calculating the torque, place the origin of the coordinate system at the center of the cube. The torque is the cross product of the force acting on the cube and the displacement from the center of the cube.

   If it is required that the net torque in any direction should be zero, what is the condition on $T_{ij}$?

3. Show that:

   (a) $\nabla \times \nabla\phi = 0$ Interpret this in terms of contours of the scalar function $\phi$.

   (b) $\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla\cdot\mathbf{u}) - \nabla^2\mathbf{u}$. (Hint: Prove that $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ and use this result.)

   (c) $S_{ij}A_{ij} = 0$ where $S_{ij}$ and $A_{ij}$ are a symmetric and antisymmetric matrix respectively.

   (d) An antisymmetric tensor $A_{ij}$ may be written as $A_{ij} = \epsilon_{ijk}\omega_k$ where $\omega_k = (1/2)\epsilon_{klm}A_{lm}$.

4. Let $f(r)$ be any scalar function of the magnitude $r = |\mathbf{r}|$ of the position vector $\mathbf{r}$ relative to the center of a sphere.

   (a) Evaluate the integral:

   $$\int_V dV\, f(r)\, \mathbf{rr} \tag{7.61}$$

   over the volume of a spherical sector with angle $\theta_0$ and axis in the $\mathbf{a}$ direction. (Hint: After integrating, what vectors or tensors can the result depend on?)

   (b) What is the result when the sector is the entire sphere?

   (Express your result in terms of integrals over the radius r).

5. Consider the parabolic flow of a fluid in a tube of radius $R$, with the velocity given in cylindrical coordinates by:

   $$v_z = V\left(1 - \frac{r^2}{R^2}\right) \tag{7.62}$$

   Separate the rate of deformation tensor into its elementary components.

6. The velocity profile of a fluid in cylindrical coordinates $(r, \phi, z)$ is given by

   $$v_\phi = \frac{\Omega}{r} \tag{7.63}$$

   where $\Omega$ is a constant, and the velocity is independent of the $z$ coordinate. This flow appears to be rotational.

   - Calculate the symmetric and antisymmetric parts of the deformation tensor. Do this in Cartesian coordinates, and in cylindrical coordinates. Are they different? Why?
   - Calculate the vorticity $\omega = \nabla \times \mathbf{v}$ for this flow. Can you explain the results?
   - What is the vorticity at the origin?

7. Consider a two-dimensional co-ordinate system, $(\xi, \eta)$, where

$$\xi = (x^2 - y^2)$$

$$\eta = 2xy$$

where $x$ and $y$ are the co-ordinate axes in a two-dimensional Cartesian co-ordinate system.

   (a) Find the unit vectors $\mathbf{e}_\xi$ and $\mathbf{e}_\eta$ in terms of the unit vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ and the co-ordinates $x$ and $y$ in the Cartesian co-ordinate system.

   (b) Is the co-ordinate system $(\xi, \eta)$ orthogonal? [1]

   (c) Express the differential displacements $d\xi$ and $d\eta$ in terms of $dx$, $dy$, $x$ and $y$.

   (d) Use the above results to express $\mathbf{e}_x$ and $\mathbf{e}_y$ in terms of $\mathbf{e}_\xi$ and $\mathbf{e}_\eta$, as well as to express $dx$ and $dy$ in terms of $d\xi$ and $d\eta$.

   (e) Insert the above results into the expression for a differential displacement,

$$d\mathbf{x} = dx\mathbf{e}_x + dy\mathbf{e}_y$$

   and obtain the differential displacement in terms of $d\xi$ and $d\eta$. Use this to obtain the scale factors $h_\xi$ and $h_\eta$ in terms of $x$ and $y$.

   (f) Can you obtain the scale factors in terms of $\xi$ and $\eta$?

   (g) Using this, obtain expressions for the divergence $\nabla.$ and the Laplacian $\nabla^2$ in the $(\xi, \eta)$ co-ordinate system.

8. In the *elliptical* co-ordinate system, the co-ordinates $r$ and $\phi$ are related to the Cartesian co-ordinates $x$ and $y$ by,

$$x = r\left(1 + \frac{\lambda^2}{r^2}\right)\cos(\phi)$$

$$y = r\left(1 - \frac{\lambda^2}{r^2}\right)\sin(\phi)$$

For this co-ordinate system,

(a) Derive expressions for the gradients of the $r$ and $\phi$ co-ordinates in terms of the unit vectors in the Cartesian co-ordinate system. *Do not try to invert the above expressions to get $r$ and $\phi$ in terms of $x$ and $y$.*

(b) Solve these to obtain the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\phi$.

(c) Is the $(r, \phi)$ co-ordinate system an orthogonal co-ordinate system?

(d) Derive expressions for the gradient and Laplacian in the elliptical co-ordinate system.