Chapter 8

Equations of motion

The governing equations for the fluid motion are of two types - the 'conservation' equations which are common for all fluids, and the 'constitutive' equations which depend on the nature of the fluid. The conservation equations are a consequence of the fact that mass, momentum and energy are conserved for all materials, while the constitutive equations give some information about the type of motion that is generated due to the forces acting on a volume element of the material. For example, for a rigid solid the conservation equations state that the mass is conserved, while the rate of change of velocity is equal to the applied force. The constitutive relation requires that the distance between any two material points is conserved during the motion, or that the deformation of the solid is zero regardless of the deforming torque that is applied on it. For a fluid, the relations are more complicated because the shape of a volume of fluid can change due to the applied forces, and the conservation and constitutive equations are also more complex. To simplify the derivation of the conservation equations, we require an integral theorem called the 'Leibnitz theorem.'

8.1 Leibnitz theorem

The Leibnitz theorem is useful for calculating the time derivative of a property in a given volume when the shape of the volume itself is changing with time, as shown in figure 8.1,

$$I(V(t),t) = \int_{V(t)} dV f(\mathbf{x},t) \tag{8.1}$$

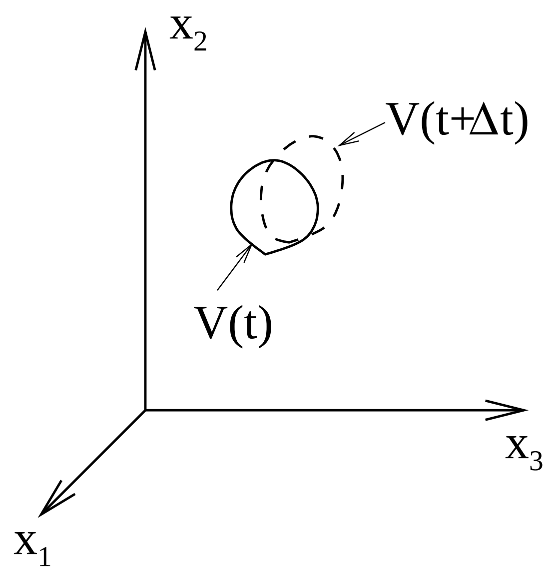

Figure 8.1: A moving volume element of fluid.

For example, the total mass within a volume is given by

$$m(V(t),t) = \int_{V(t)} dV\,\rho(\mathbf{x},t) \tag{8.2}$$

and the mass conservation within a moving volume of fluid requires that

$$\frac{dm(V(t),t)}{dt} = 0 \tag{8.3}$$

It should be noted that the mass conservation equation cannot be written as

$$\int_{V(t)} dV\,\frac{\partial\rho}{\partial t} = 0 \tag{8.4}$$

because the partial derivative is evaluated at a fixed point in space, whereas the volume $V(t)$ is moving.

The Leibniz rule can be best understood considering its one dimensional analogue. Consider the one dimensional integral of a function shown in figure 8.2,

$$I(t) = \int_{a(t)}^{b(t)} dx\, f(x,t) \tag{8.5}$$

Here, note that the limits of integration are functions of time. When we want to calculate the time derivative of this, it is necessary to take into account the movement of the limits of integration.

$$
\begin{aligned}
\tfrac{dI(t)}{dt} &= \lim_{\Delta t\to 0}\frac{\Delta I}{Dt} \\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{a_2(t+\Delta t)}^{a_1(t+\Delta t)} dx\, f(x,t+\Delta t) - \int_{a_2(t)}^{a_1(t)} dx\, f(x,t)\right]
\end{aligned}
\tag{8.6}
$$

Using diagrammatic representation, it can be shown that this is equal to:

$$
\begin{aligned}
\tfrac{dI(t)}{dt} &= \lim_{\Delta t\to 0}\left[\int_{a_2(t)}^{a_1(t)} dx\,\frac{f(x,t+\Delta t)-f(x,t)}{\Delta t} + f(a_1)\frac{a_1(t+\Delta t)-a_1(t)}{\Delta t} - f(a_2)\frac{a_2(t+\Delta t)-a_2(t)}{\Delta t}\right] \\
&= \int_{a_2(t)}^{a_1(t)} dx\,\partial_t f(x,t) + f(a_1)d_t a_1 - f(a_2)d_t a_2
\end{aligned}
\tag{8.7}
$$

Note that $d_t a_1$ and $d_t a_2$ are the velocities of the limits of the integral. Therefore, the above integral can also be written as:

$$
\frac{dI(t)}{dt} = \int_{a(t)}^{a_1(t)} dx\,\partial_t f(x,t) + \sum_l n_l v_l f(a_l)
\tag{8.8}
$$

where v_l are the boundaries of the limits a_l, and n_l is the direction of the outward normal to the limit of integration, which is -1 for the lower limit and $+1$ for the upper limit. The Liebnitz theorem is the three dimensional analogue of the above equation over a volume $V(t)$ that is a function of time. If we consider an integral over a volume which is changing in time:

$$
I(t) = \int_{V(t)} dV\, f(x_i,t)
\tag{8.9}
$$

then the time derivative of this integral is given by:

$$
\frac{dI(t)}{dt} = \int_{V(t)} dV\,\partial_t f(x_i,t) + \int_A dA\, n_i v_i f
\tag{8.10}
$$

where n_i is the normal to the surface. The above expression can be simplified using the divergence theorem:

$$
\frac{dI(t)}{dt} = \int_{V(t)} dV\,\partial_t f(x_i,t) + \int_{V(t)} dV\,\partial_i(v_i f)
\tag{8.11}
$$

This is the most general form of the Liebnitz rule.

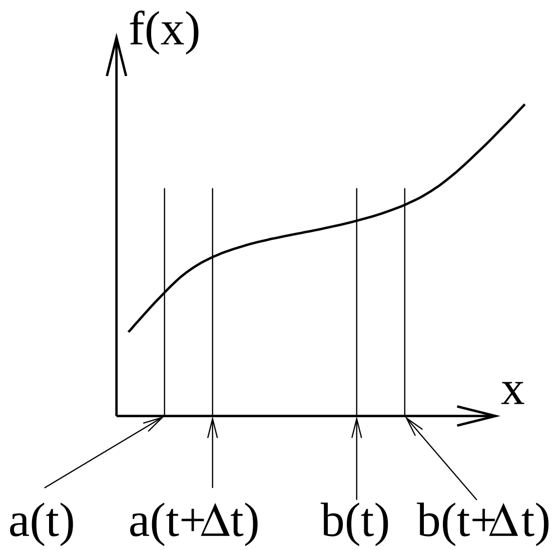

Figure 8.2: Leibniz rule in one dimension.

8.2 Mass conservation

The law of conservation of mass states that in any volume of fluid in the flow, the total mass is conserved. If $\rho(x_i, t)$ is the density of the fluid at a point x_i and time t, then the total mass in a control volume $V(t)$ is:

$$M = \int_{V(t)} dV \rho(x_i, t) \tag{8.12}$$

The law of conservation of mass states the total mass in a differential volume is invariant in time.

$$\frac{dM}{dt} = \frac{d}{dt} \int V(t) dV \; \rho(x_i, t) = 0 \tag{8.13}$$

This can be simplified using the Liebnitz rule:

$$\int_{V(t)} dV \; [\partial_t \rho(x_i, t) + \partial_i(v_i \rho(x_i, t)] = 0 \tag{8.14}$$

This is true for every differential volume in the fluid, and therefore the integrand in the above equation has to be zero:

$$\partial_t \rho + \partial_i(\rho v_i) = \frac{D\rho}{Dt} + \rho(\partial_i v_i) = 0 \tag{8.15}$$

where the second equality follows from the definition of the total derivative. If we consider an *incompressible fluid*, where the density $\rho(x_i, t)$ is not a function of position or time, then the mass conservation equation reduces to the following simple form:

$$\partial_i v_i = 0 \tag{8.16}$$

8.3 Momentum conservation

The momentum conservation equation is a consequence of Newton's second law of motion, which states that the rate of change of momentum in a differential volume is equal to the total force acting on it. The momentum in a volume $V(t)$ is given by:

$$P_i = \int_{V(t)} dV\ \rho v_i \tag{8.17}$$

and Newton's second law states that:

$$\frac{dP_i}{dt} = \int_{V(t)} dV\ \left[\partial_t(\rho v_i) + \partial_j(\rho v_i v_j)\right] = \text{Sum of forces} \tag{8.18}$$

The forces acting on a fluid are usually of two types. The first is the body force, which act on the bulk of the fluid, such as the force of gravity, and the surface forces, which act at the surface of the control volume. If F_i is the body force per unit mass of the fluid, and R_i is the force per unit area acting at the surface, then the momentum conservation equation can be written as:

$$\int_{V(t)} dV\ \left[\partial_t(\rho v_i) + \partial_j(\rho v_i v_j)\right] = \int_{V(t)} dV\ \rho F_i + \int_S dS R_i \tag{8.19}$$

where S is the surface of the volume $V(t)$.

In order to complete the derivation of the equations of motion, it is necessary to specify the form of the body and surface forces. The form of the body forces is well known – the force due to gravity per unit mass is just the gravitational acceleration g, while the force due to centrifugal acceleration is given by ωR^2 where ω is the angular velocity and R is the radius. The form of the surface force is not well specified, however, and the 'constitutive equations' are required to relate the surface forces to the motion of the fluid. But before specifying the nature of the surface forces, one can derive some of the properties of the forces using symmetry considerations.

The force acting on a surface will, in general, depend on the orientation of the surface, and will be a function of the unit normal to the surface, in addition to the other fluid properties:

$$R_i = R_i(n_i) \tag{8.20}$$

We can use symmetry considerations to show that the force is a linear function of the unit normal to the surface. If we consider a surface dividing the fluid at a point P, then the force acting on the two sides of the surface have to be equal according to Newton's third law. Since the unit normal to the two sides are just the negative of each other, this implies that:

$$R_i(-n_i) = -R_i(n_i) \tag{8.21}$$

This suggests that the surface force could be a linear function of the normal:

$$R_i = T_{ij} n_j \tag{8.22}$$

That this is the case can be shown using Cauchy's theorem. Consider a tetrahedron with three surfaces located along the three coordinate axes, with lengths Δx_1, Δx_2 and Δx_3, as shown in figure 8.3. In the limit as three lengths go to zero, the density and velocity in the region can be taken as constants. The integral momentum conservation equation then becomes:

$$\begin{aligned}[\partial_t(\rho v_i) + &\ \partial_j(\rho v_i v_j)]\Delta x_1\,\Delta x_2\,\Delta x_3 = [\rho F_i]\Delta x_1\,\Delta x_2\,\Delta x_3 \\ &+ R_i(n_i)\Delta S + R_i(-e_1)\Delta S_1 + R_i(-e_2)\Delta S_2 + R_i(-e_3)\Delta S_3\end{aligned} \tag{8.23}$$

In the limit as the length of the sides Δx_i go to zero, the inertial terms and the body forces become much smaller than the body forces, we can write:

$$R_i(n_i)\Delta S = R_i(e_1)\Delta S_1 + R_i(e_2)\Delta S_2 + R_i(e_3)\Delta S_3 \tag{8.24}$$

The areas along the three axes are related to the area of the triangular plane as:

$$\Delta S_i = n_i \Delta S \tag{8.25}$$

Using this relation, we get:

$$R_i(n_i) = n_1 R_i(e_1) + n_2 R_i(e_2) + R_i(e_3) \tag{8.26}$$

This is exactly of the same form as:

$$R_i(n_i) = T_{ij} n_j \tag{8.27}$$

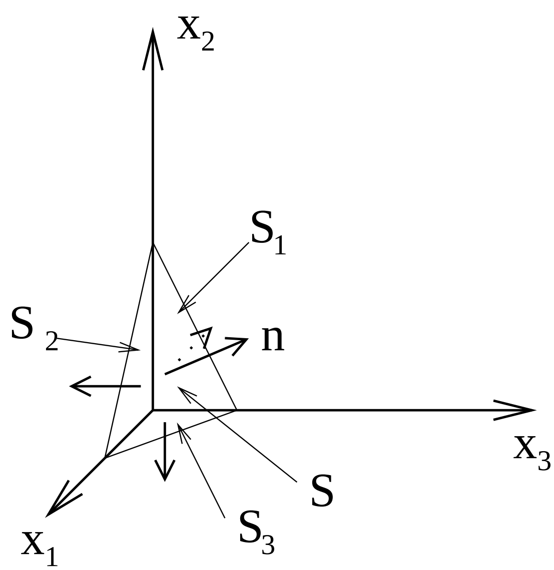

Figure 8.3: Forces acting on a tetrahedral control volume.

The tensor T_{ij} is called the 'stress tensor', and this depends on the position in the fluid but not on the direction of the unit normal.

Using the stress tensor, the conservation equation is given by:

$$\int_{V(t)} dV \; [\partial_t(\rho v_i) + \partial_j(\rho v_i v_j)] = \int_{V(t)} dV \; \rho F_i + \int_S dS T_{ij} n_j \tag{8.28}$$

This can be further simplified using the divergence theorem:

$$\int_{V(t)} dV \; [\partial_t(\rho v_i) + \partial_j(\rho v_i v_j)] = \int_{V(t)} dV \; [\rho F_i + \partial_j T_{ij}] \tag{8.29}$$

Since the above equation is valid for any differential volume in the fluid, the integrand must be equal to zero:

$$\partial_t(\rho v_i) + \partial_j(\rho v_i v_j) = \rho F_i + \partial_j T_{ij} \tag{8.30}$$

The above equation can be simplified using the mass conservation equation:

$$\rho \partial_t v_i + \rho v_j \partial_j v_i = \rho F_i + \partial_j T_{ij} \tag{8.31}$$

8.4 Angular momentum conservation

It turns out that the angular momentum conservation equation gives us just one additional piece of information that cannot be derived from the linear momentum equation, i.e. that the stress tensor T_{ij} is symmetric. The angular momentum of a volume of fluid is given by:

$$L = \int_{V(t)} dV\, \mathbf{x} \times \rho \mathbf{v} = \int_{V(t)} dV\, \epsilon_{ijk} x_j \rho v_k \tag{8.32}$$

The angular momentum conservation equation states that:

$$\frac{dL}{dt} = \text{Sum of torques} \tag{8.33}$$

The torque due to the body forces on a volume of fluid is given by:

$$T_{bi} = \int_{V(t)} dV\, \epsilon_{ijk} x_j \rho g_k \tag{8.34}$$

where g_k is the acceleration due to gravity, while the torque due to the surface forces is given by:

$$T_{si} = \int_{V(t)} dV\, \epsilon_{ijk} x_j \partial_l T_{kl} \tag{8.35}$$

The above expression can be simplified to read:

$$\begin{aligned} T_{si} = & \int_S dS\, [\epsilon_{ijk} x_j T_{kl} n_l) \\ &= \int_{V(t)} dV \partial_l [\epsilon_{ijk} x_j T_{kl}] \\ &= \int_{V(t)} dV\, [\epsilon_{ijk} x_j \partial_l T_{kl} + \epsilon_{ijk} T_{kj}] \end{aligned} \tag{8.36}$$

The rate of change of angular momentum in a differential volume is given by:

$$\begin{aligned} \tfrac{dL}{dt} &= \frac{d}{dt} \int_{V(t)} dV\, \epsilon_{ijk} x_j \rho v_k \\ &= \int_{V(t)} dV\, \left[\epsilon_{ijk} x_j \partial_t(\rho v_k) + \partial_l(\epsilon_{ijk} x_j \rho v_k v_l\right] \\ &= \int_{V(t)} dV\, \left[\epsilon_{ijk} x_j \partial_t(\rho v_k) + \epsilon_{ijk} \delta_{jl} \rho v_k v_l + \epsilon_{ijk} x_j \partial_l(\rho v_k v_l)\right] \\ &= \int_{V(t)} dV\, \epsilon_{ijk} x_j [\partial_t(\rho v_k) + \partial_l(\rho v_k v_l)] \end{aligned} \tag{8.37}$$

Here, $\epsilon_{ijk}\delta_{jl}\rho v_k v_l = \epsilon_{ijk}\rho v_j v_k = 0$ because ϵ_{ijk} is an antisymmetric tensor and $v_k v_l$ is a symmetric tensor. Equating the rate of change of angular momentum and the applied torques, and discarding the volume integrals, we find that:

$$\epsilon_{ijk}x_j\{\partial_t(\rho v_k + \partial_l(\rho v_k v_l) - F_k - \partial_l T_{kl}\} = \epsilon_{ijk}T_{kj} \tag{8.38}$$

The term in the flower brackets is just equal to the linear momentum equation, which is zero. This implies that $\epsilon_{ijk}T_{kj}$ is zero, or T_{ij} is a symmetric matrix. Therefore, the angular momentum conservation equation has given us an additional piece of information about the stress tensor.

8.5 Constitutive equations for the stress tensor

Just as we had earlier separated the strain rate tensor into an antisymmetric traceless part, a symmetric part and an isotropic part, it is conventional to separate the stress tensor into an isotropic part and a symmetric traceless 'deviatoric' part:

$$T_{ij} = -p\delta_{ij} + \tau_{ij} \tag{8.39}$$

where $\tau_{ii} = 0$, and p is the 'pressure' in the fluid. In the absence of fluid flow, the deviatoric part of the stress tensor becomes zero. This is because if the deviatoric part is not zero, then the fluid volume will get distorted in the absence of flow and the fluid cannot be at equilibrium. The pressure field is related to the local density of the system by a thermodynamic equation of state.

The shear stress, τ_{ij}, is a function of the fluid velocity. However, the stress cannot depend on the fluid velocity itself, because the stress has to be invariant under a 'Galilean transformation', i.e., when the velocity of the entire system is changed by a constant value. Therefore, the stress has to depend on the gradient of the fluid velocity. In a 'Newtonian fluid', we make the assumption that the stress is a linear function of the velocity gradient. In general, the linear relation can be written as:

$$\tau_{ij} = \mu_{ijkl}\partial_k v_l \tag{8.40}$$

where μ_{ijkl} is a fourth order tensor. This tensor is a property of the fluid. In an isotropic fluid, there is no preferred direction in space, and the tensor

μ_{ijkl} can only be a function of the isotropic tensor δ_{ij} and ϵ_{ijk}. The most general tensor that can be constructed from these is:

$$\mu_{ijkl} = A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + C\delta_{il}\delta_{jk} \tag{8.41}$$

However this tensor has to satisfy certain symmetry properties. Since τ_{ij} is symmetric, we require that $\mu_{ijkl} = \mu_{jikl}$, which in turn implies that $B = C$. In addition, $\mu_{iikl} = 0$ since $\tau_{ii} = 0$. This implies that:

$$A\delta_{ii}\delta_{kl} + 2B\delta_{ik}\delta_{il} = 0 \tag{8.42}$$

or $A = (-2B/3)$. Using this, we get the expression for the deviatoric stress (shear stress) in a Newtonian fluid:

$$\tau_{ij} = \mu(\partial_i v_j + \partial_j v_i - (2/3)\delta_{ij}\partial_k v_k) \tag{8.43}$$

where μ is called the 'coefficient of viscosity' of the fluid. This can also be expressed in terms of the rate of strain tensor:

$$\tau_{ij} = 2\mu e_{ij} \tag{8.44}$$

where e_{ij} is the symmetric traceless part of the rate of deformation tensor.

The above constitutive equation was derived for a Newtonian fluid with the assumption that the shear stress is a linear function of the strain rate. However, the stress could be a non - linear function of the strain rate in complex fluids such as polymers solutions.

$$\tau_{ij} = f(e_{ij}) \tag{8.45}$$

Since e_{ij} is a frame indifferent tensor, any tensor that can be written as:

$$\tau_{ij} = \text{scalar} \times e_{ij} \tag{8.46}$$

satisfies the conditions of frame indifference. There are three frame indifferent scalars that can be constructed from the tensor e_{ij}:

$$I_1 = s_{ii} \quad I_2 = s_{ij}s_{ji} \quad I_3 = \mathrm{Det}(s_{ij}) \tag{8.47}$$

Therefore, any constitutive equation of the form:

$$\tau_{ij} = \mu(I_1, I_2, I_3)e_{ij} \tag{8.48}$$

would satisfy the requirements of material frame indifference.

The constitutive equation for the stress tensor can be inserted into the momentum conservation equation to obtain:

$$\begin{aligned} \partial_t \rho + \partial_i(\rho v_i) &= 0 \\ \rho \partial_t v_i + \rho v_j \partial_j v_i &= -\partial_i p + \mu(\partial_j^2 v_i - (2/3)\partial_i \partial_j v_j) \end{aligned} \tag{8.49}$$

These are the Navier - Stokes mass and momentum conservation equations. For an incompressible fluid, where the density is a constant in both space and time, the Navier - Stokes equations have a particularly simple form:

$$\begin{aligned} \partial_i v_i &= 0 \\ \partial_t v_i + v_j \partial_j v_i &= -\rho^{-1} \partial_i p + \nu \partial_j^2 v_i \end{aligned} \tag{8.50}$$

where $\nu = (\mu/\rho)$ is the kinematic viscosity.

8.6 Energy conservation

The equation for the mechanical energy balance can be derived quite easily by multiplying the momentum conservation equation by v_i and doing some integration by parts:

$$\partial_t[\rho v_i^2/2] + \partial_i[v_i(\rho v_j^2/2)] = -v_i \partial_i p + v_i \partial_j \tau_{ji} + \rho v_i F_i \tag{8.51}$$

The integral energy balance equation over a volume V can be written as:

$$\begin{aligned} \textstyle\int_V dV \{\partial_t[\rho v_i^2/2] + \int_V dV \partial_i[v_i(\rho v_i^2/2)] &= \int_V [-v_i \partial_i p + v_i \partial_j \tau_{ij} + \rho v_i F_i] \\ &= \int_V dV[-\partial_i(v_i p) + p \partial_i v_i + \partial_j(v_i \tau_{ij} - \tau_{ij} \partial_j v_i + \rho v_i F_i] \end{aligned} \tag{8.52}$$

The terms on the right side of the equation can be simplified as follows. The first term can be converted into a surface integral:

$$\int_V dV[-\partial_i(v_i p)] = \int_S dS(-v_i p) \tag{8.53}$$

This is the rate of change of energy due to the pressure forces acting at the surface of the volume V. The second term can be simplified using the mass conservation equation:

$$p \partial_i v_i = -p \rho^{-1}(\partial_t \rho + v_j \partial_j \rho) = -p \rho^{-1} \frac{D\rho}{Dt} \tag{8.54}$$

Therefore, this term represents the increase in the energy of the fluid due to the compression which the fluid experiences. The third term can once again be expressed as a volume integral:

$$\int_V dV \partial_i (v_j \tau_{ij}) = \int_S dS n_i v_j \tau_{ij} \tag{8.55}$$

This represents the rate of change of energy due to the work done by the shear stress at the surface. The fifth term is, quite obviously, the work done by the body forces. It remains to interpret the fourth term in the above equation. This is given by:

$$\begin{aligned} \mathcal{D} &= (2\mu s_{ij} - (2/3)\mu\delta_{ij}s_{kk})\partial_i v_j \\ &= (2\mu s_{ij} - (2/3)\mu\delta_{ij}s_{kk})s_{ij} \\ &= 2\mu s_{ij}s_{ji} - (2/3)\mu s_{kk}^2 \end{aligned} \tag{8.56}$$

This represents the rate at which energy is dissipated in a volume of fluid due to the viscous stresses. It can be shown that the rate of dissipation is always positive,

$$\begin{aligned} \mathcal{D} &= 2\mu s_{ij}s_{ji} - (2/3)\mu s_{kk}^2 \\ &= 2\mu(s_{11}^2 + s_{22}^2 + s_{33}^2 + 2s_{12}s_{21} + 2s_{13}s_{31} + 2s_{23}s_{32}) \\ &\quad -(2/3)\mu(s_{11}^2 + s_{22}^2 + s_{33}^2 + 2s_{11}s_{22} + 2s_{11}s_{33} + 2s_{22}s_{33}) \\ &= 2\mu(s_{12}^2 + s_{13}^2 + s_{23}^2) + (2\mu/3)((s_{11} - s_{22})^2 + (s_{22} - s_{33})^2 + (s_{11} - s_{33})^2) \end{aligned} \tag{8.57}$$

8.7 Boundary conditions

The Navier - Stokes equations of motion are partial differential equations, and in order to solve the equations in a volume it is necessary to specify the boundary conditions at the bounding surface of the volume. Note that the surfaces could, in general, be at infinity, in which case it is necessary to specify the boundary conditions in the limit when the coordinate tends to infinity. There are two types of boundary conditions - the velocity conditions and the stress conditions.

The most commonly used velocity boundary condition is that the velocity vector is continuous across the material interface. This is called the no - slip condition. In effect, this implies that the velocity of a fluid particle in one fluid at the interface is equal to that of a particle in the other fluid. In the case of solid boundaries, it is assumed that the velocity of the fluid at the surface is equal to the velocity of the solid surface itselr. This is physically reasonable, because if their velocities are different, we would expect a viscous shear stress to act at the interface. This would imply an infinite acceleration since the interface has negligible volume. However, there are certain extraordinary conditions where it may be necessary to use a slip boundary condition at an interface. In the case of a gas - liquid interface, it is usually not necessary to specify velocity boundary conditions; it turns out that the stress boundary conditions are sufficient to completely specify the problem.

There are two types of stress conditions - the normal stress and the shear stress boundary conditions. The shear stress acts tangential to the surface, and the shear stress at a surface with normal n_i is given by:

$$(\delta_{ij} - n_i n_j)\tau_{jk} n_k \tag{8.58}$$

Note that the operator $(\delta_{ij} - n_i n_j)$ is the 'tangential' operator, and when this acts on a vector f_j it subtracts the normal component of the vector since:

$$(\delta_{ij} - n_i n_j) n_j = 0 \tag{8.59}$$

The tangential stress boundary condition requires that the stress is continuous across a material interface. This is because if we consider a small differential volume of thickness Δ and L at the interface. Let $R_i(x_i)$ be the surface stress (force per unit area) and $F_i(x_i)$ be the body force (the force per unit volume) due to the body force and the inertial forces. Then the total force due to the surface stresses and the inertial and body forces on this differential volume are given by:

$$\begin{gathered} \text{Force due to surface stresses} = R_i L^2 \\ \text{Force due to the inertial and body forces} = F_i L^2 \Delta \end{gathered} \tag{8.60}$$

In the limit $\Delta \rightarrow 0$, the force due to the inertial and body forces becomes negligible, and therefore the force balance equation contains just the force due to the surface stresses. Therefore, the stress balance condition requires that the difference in the tangential force due to the surface stresses is zero across the interface:

$$(\delta_{ij} - n_i n_j)(\tau^1_{jk} - \tau^2_{jk}) n_k = 0 \tag{8.61}$$

Note that the pressure p does not enter into the above equation because it gives rise to a normal force.

The normal force balance condition contains an additional term due to the 'surface tension', which is defined as the force exerted per unit length of the interface. The force due to surface tension per unit area of the interface is given by:

$$F_{sti} = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) n_i \tag{8.62}$$

where γ is the coefficient of surface tension, and R_1 and R_2 are defined as the principal radii of curvature, which are the radii of curvature along any two orthogonal planes intersecting the surface. Here, the radii of curvature are considered to be positive if they are located in the medium into which n_i is directed. Moreover, it is known from analytical geometry that for any continuous surface, the sum of the inverse of the principal radii of curvature along any two orthogonal axes is a constant. Therefore, the force due to the surface tension is unique regardless of the directions along which the two principal radii R_1 and R_2 are measured. The normal stress balance condition then becomes:

$$n_i T^1_{ij} n_j - n_i T^2_{ij} n_j = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \tag{8.63}$$

Note that in the above equation, the center of curvature of the two radii R_1 and R_2 are located in medium 1.

There are many cases where the principal radii of curvature are small compared to the other length scales in the problem. In these cases, the sum of the principal radii of curvature can be related to the divergence in the unit normal to the surface. Consider a coordinate system $\{x_1, x_2\}$ located at an interface. The divergence of the unit normal is given by:

$$\begin{aligned} \partial_i n_i &= \frac{\partial n_1}{\partial x_1} + \frac{\partial n_2}{\partial x_2} \\ &= \lim_{\Delta x_1, \Delta x_2 \to 0} \frac{\Delta n_1}{\Delta x_1} + \frac{\Delta n_2}{\Delta x_2} \end{aligned} \tag{8.64}$$

By geometrical construction, we can show that $(\Delta n_1/\Delta x_1) = (1/R_1)$ and $(\Delta n_2/\Delta x_2) = (1/R_2)$, and therefore:

$$\partial_i n_i = \frac{1}{R_1} + \frac{1}{R_2} \tag{8.65}$$

With this, the normal stress balance condition becomes:

$$n_i T^1_{ij} n_j - n_i T^2_{ij} n_j = \gamma \partial_i n_i \tag{8.66}$$

The stress boundary conditions derived above are directly applicable to the interface between two fluids. In the case of a solid - fluid interface, it is usually not necessary to use stress boundary condition; the velocity boundary conditions usually suffice. This is because a rigid solid is defined as one which does not deform regardless of the stress applied at the surface, and the shear and normal forces due to the fluid are balanced by equal and opposite forces generated by the solid itself. Therefore, stress boundary conditions at the interface between a fluid and a solid are unnecessary. In the case of a liquid - gas interface, we can make use of the fact that the viscosity of a gas is usually much smaller (about 10^{-3} times smaller) than the viscosity of a fluid. Therefore, the visous stresses in the gas are also much smaller than that of the fluid, and they can be neglected. Since the shear stress is caused entirely due to viscous effects, the shear stress balance condition becomes:

$$(\delta_{ij} - n_i n_j)\tau^l_{jk} n_k = 0 \tag{8.67}$$

where τ^l_{jk} is the deviatoric stress in the liquid. The normal stress in the gas is entirely due to the pressure, since the component due to viscous effects has been neglected. The normal stress condition then becomes:

$$n_i T^l_{ij} n_j - p^g = \gamma \partial_i n_i \tag{8.68}$$

where the unit normal n_i is directed into the liquid.

8.8 Solution of Problems

The Navier - Stokes equation, which are the basic equations of fluid dynamics, are non - linear partial differential equations. As a result, there is no general technique for solving these equations. In practical situations one can make certain simplifications which may render these equations soluble. However, before making these approximations, certain systemmatic steps do help. The first is to choose a coordinate system that simplifies the problem. The coordinate systems that we will deal with in the present course are the Cartesian, cylindrical and spherical coordinate systems. However, in certain systems more complex coordinate systems may be necessary, and it is necessary to have a more complete knowledge of analytical geometry to solve these problems.

Further, it is necessary to non - dimensionalise the velocities, distances and time scales in the problem correctly. Often, the choice of correct length, time and velocity scales could be subjective, especially in 'singular perturbation' problems, and it is necessary to have some experience to do it correctly. Hopefully the examples that we will work out here will provide some experience in this. Incorrect scaling often leads to inconsistent results, and the appearance of inconsistencies can often be taken as an indication that there is something wrong with the scaling. Another step that could help, related to scaling, is to understand clearly the forces that 'drive' the flow, and to visualise the paths followed by the fluid particles. The qualitative picture can be used to clearly and unambiguously scale the variables in the problem.

Finally, after obtaining a sufficiently simplified form of the governing equations, it is necessary to ensure that one has a sufficient number of boundary conditions so that the problem is completely specified. For an n^{th} order ordinary differential equation, one requires a total of n boundary condition, and for a partial differential equation in contains n derivatives along the x_i coordinate one requires n boundary conditions in that direction. In this course, we will discuss a number of techniques for simplifying and solving the equations of motion for the fluid.

The exactly solvable problems in fluid dynamics are usually of one of the following types:

1. One dimensional problems, where the fluid flow is along one direction only. Flows in channels and tubes are of this type.

2. Low Reynolds number flows, where the inertial terms in the conservation equation can be neglected. In this case, the equations of motion become linear, and several simplification procedures can be used. Flows of colloidal suspensions, slurries and many flows in chemical engineering applications which are at relatively low speeds are of this type.

3. Inviscid flows at high Reynolds numbers, where the vicsous terms in the conservation equations can be neglected. This leads to a fairly simple momentum conservation equation – the Laplace equation for the 'velocity potential' $\nabla^2 \phi = 0$. This equation is often used to model the flows over aircraft wings and other bodies at high speeds.

4. Boundary layer flows where very different length scales are involved. In this case, asymptotic techniques can be used to simplify the problem.

5. Problems concerning the stability of a steady flow. Here, one usually uses a linear stability analysis, where small fluctuations about the

steady solution are considered, and the equations for the growth of these fluctuations are linear.

There are other flows, such as turbulent flows, where it is difficult to make simplifications. Here, it is necessary to use certain scaling arguments to make some qualitative predictions about the nature of the flow. In the next few lectures, we will solve some problems in the above catogeries using mathematical techniques such as:

1. Similarity solutions
2. Separation of variables
3. Greens functions
4. Asymptotic analysis
5. Boundary layer theory