Chapter 9

Viscous flow

The Navier-Stokes equations for an incompressible fluid in indicial notation are

$$\partial_i u_i = 0 \tag{9.1}$$

$$\rho(\partial_t + u_j \partial_j) u_i = -\partial_i p + \mu \partial_j^2 u_i \tag{9.2}$$

When these are applied to practical problems, it is convenient to scale the length and velocity scales by the characteristic length L and velocity U in the problem. For example, in the case of the flow in a pipe, the characteristic length is the pipe diameter, while the characteristic velocity is the average velocity of the fluid. In the case of a particle falling through a fluid, the characteristic length is the particle diameter and the characteristic velocity is the velocity of the particle. The scaled velocity and length are defined as $x_i^* = (x_i/L)$, and $u_i^* = (u_i/U)$. The natural time scale in the problem is then (L/U), and the scaled time is defined as $t^* = (tU/L)$. Expressed in terms of these scaled variables, the Navier-Stokes equations are

$$\partial_i^* u_i^* = 0 \tag{9.3}$$

$$\frac{\rho U^2}{L} \left(\partial_t^* + u_j^* \partial_j^* \right) u_i^* = -\frac{1}{L} \partial_i^* p + \frac{\mu U}{L^2} \partial_j^{*2} u_i^* \tag{9.4}$$

where $\partial_i^* = (\partial/\partial x_i^*)$, and $\partial_t^* = (\partial/\partial t^*)$. The pressure has not yet been scaled in equation 9.4, and we choose to define the scaled pressure as $p^* = (pL/\mu U)$, so that the pressure and viscous terms in the equation are of the same magnitude. With this, the Navier-Stokes momentum equation becomes

$$\frac{\rho U L}{\mu} \left(\partial_t^* + u_j^* \partial_j^* \right) u_i^* = -\partial_i^* p^* + \partial_j^{*2} u_i^* \tag{9.5}$$

The dimensionless number $(\rho U L/\mu)$ is the Reynolds number, which provides the ratio of convection and diffusion for the fluid flow. This can also be written as (UL/ν), where $nu = (\mu/\rho)$ is the kinematic viscosity or momentum diffusivty.

When the Reynolds number is small, the convective terms in the momentum equation 9.5 can be neglected in comparison to the viscous terms on the right side of 9.5. The Navier-Stokes equations reduce to the 'Stokes equations' in this limit,

$$\partial_i u_i = 0 \tag{9.6}$$

$$-\partial_i p + \mu \partial_j^2 u_i = 0 \tag{9.7}$$

The Stokes equations have two important properties — they are *linear*, since both the equations are linear in the velocity u_i, and they are *quasi-steady*, since there are no time derivatives in the Stokes equations. The absence of time derivatives in the governing equations implies that the velocity field due to the forces exerted on the fluid depend only on the instantaneous value of the forces, or the velocity boundary conditions at the surfaces bounding the fluid, and not the time history of the forces exerted on the fluid. For example, in order to determine the velocity field due to force exerted by a sphere falling through a fluid, it is sufficient to know the instantaneous value of the velocity of the sphere, and it is not necessary to know the details of the previous evolution of the velocity of the sphere. This is because the flow is diffusion dominated, and it is assumed that diffusion is instantaneous at zero Reynolds number. A similar feature is encountered in the solution for the diffusion equation in the limit of low Peclet number, where convective effects are neglected.

$$D\nabla^2 c = 0 \tag{9.8}$$

In this case, the concentration field is determined at any instant if the value of the boundary conditions at the bounding surfaces are know, and it is not necessary to know the previous evolution of the previous evolution of the concentration at the bounding surfaces.

Diffusion is not instantaneous in reality, of course, and there is a distance beyond which the assumption of instantaneous diffusion is not valid. For example, consider the case of a sphere settling through a fluid. Though diffusion is fast compared to convection over lengths comparable to the sphere diameter in the limit of low Reynolds number, it is expected that there is some larger length over which diffusion is not instantaneous, and convective effects become important. This 'diffusion length' L_D can be estimated by balancing the convective and diffusive terms in the momentum equation,

$$\begin{aligned} \rho u_j \partial_j u_i &\sim (\rho U^2/L_D) \\ \mu \partial_j^2 u_i &\sim (\mu U/L_D) \end{aligned} \tag{9.9}$$

A balance between these two indicates that $L_D \sim (\mu/\rho U)$, or $(L_D/L) \sim \mathrm{Re}^{-1}$. Thus, convective effects become important when the ratio of the distance from the sphere and the sphere radius is large compared to Re^{-1}.

The *linearity* of the Stokes equations has some useful consequences. For example linearity can be used to deduce that if the force exerted by a surface on a fluid is reversed, then the fluid velocity field is also exactly reversed at all points in the fluid, as shown in figure 9.1. This is called the 'Stokes flow reversibility'. Similarly, if the force exerted by the surfaces on the sphere is reduced by a factor of 2, then the fluid velocity is also reduced by a factor of 2 at all points. In addition, linear superposition can be used to separate a problem into smaller sub-problems, and then add up the results. For example, consider a sphere moving in a fluid as shown in figure 9.2. This problem can be separated into three sub-problems, each of which consists of a sphere which has a velocity along one of the coordinate axes, with the stipulation that the sum of the three velocity vectors is equal to the velocity of the sphere in the original problem. The velocity fields in each of these three sub-problems are calculated, and are added up in order to obtain the velocity field in the original problem.

9.1 Spherical harmonic solutions

The Stokes equations can be written as two Laplace equations for the pressure and the velocity fields by taking the divergence of the momentum equation 9.7.

$$\begin{aligned} &\partial_i(-\partial_i p + \mu \partial_j^2 u_i) \\ &\quad = -\partial_i^2 p + \mu \partial_j^2 \partial_i u_i \\ &\quad = -\partial_i^2 p \end{aligned} \tag{9.10}$$

Equation 9.10 indicates that the Laplacian of the pressure field is identically zero,

$$\partial_i^2 p = 0 \tag{9.11}$$

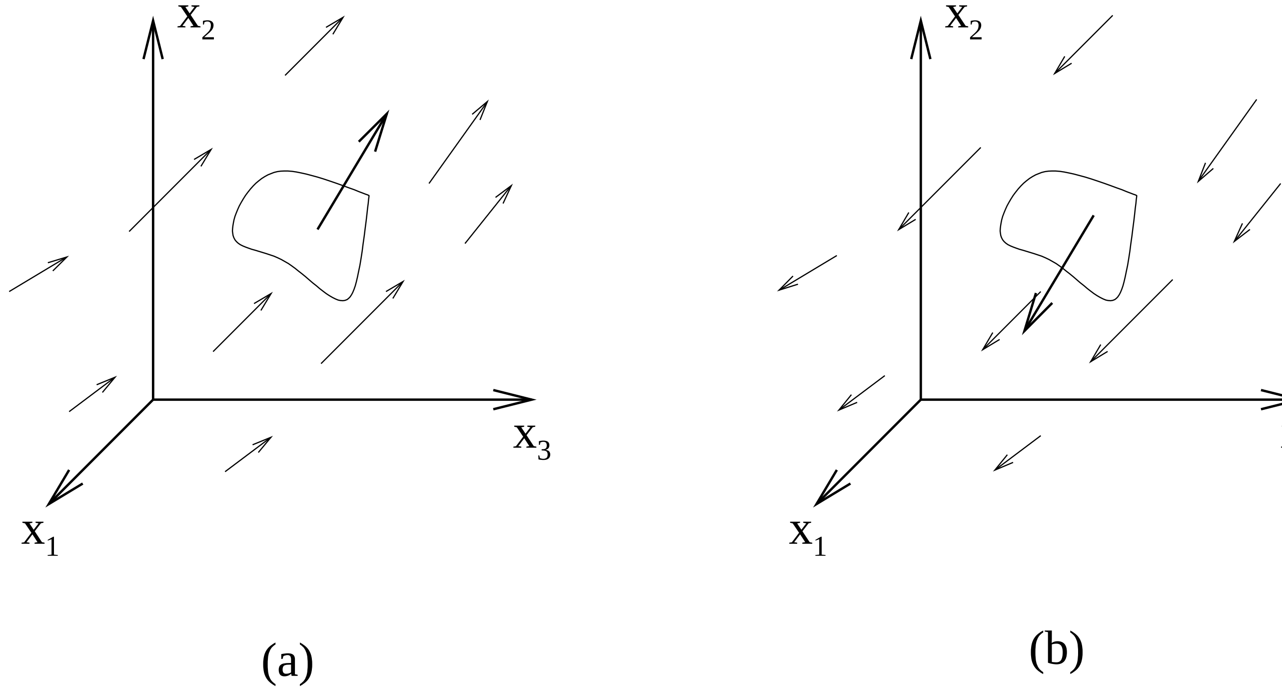


Figure 9.1: When the force exerted on the fluid is reversed, the fluid velocity is reversed at all points in Stokes flow.

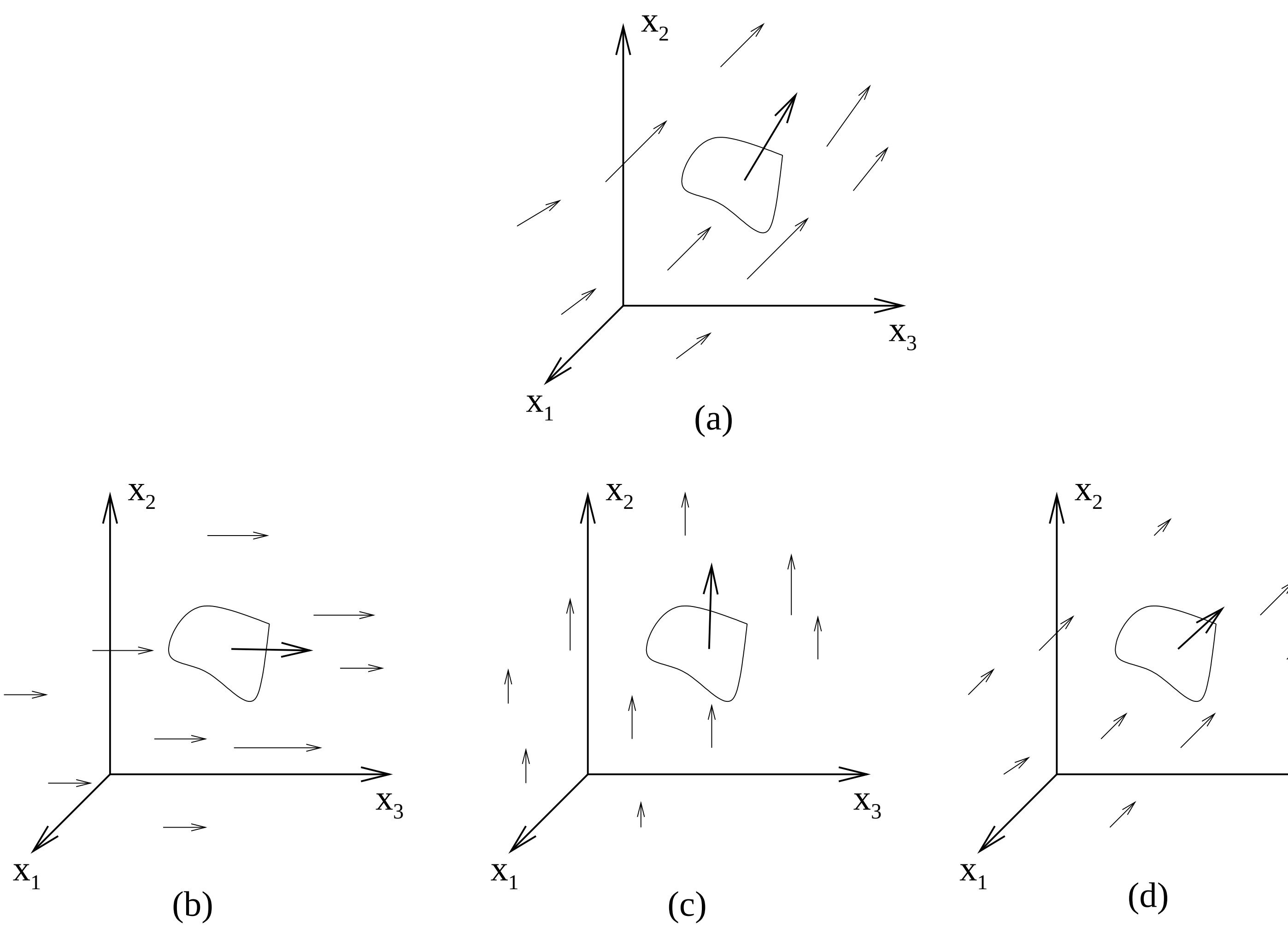


Figure 9.2: Use of linear superposition for separating the problem into simpler sub-problems.

The velocity field in the momentum equation can be separated into two parties, a 'homogeneous solution' which satisfies

$$\partial_i^2 u_i^{(h)} = 0 \tag{9.12}$$

and a particular solution which satisfies

$$\mu \partial_i^2 u_i^{(p)} = \partial_i p \tag{9.13}$$

The particular solution is any one solution to the equation 9.13. One solution which satisfies equation 9.13 is

$$u_i^{(p)} = \frac{1}{2\mu} p x_i \tag{9.14}$$

This solution can be verified as follows. If we take the gradient of the particular solution 9.14, we get

$$\partial_j u_i^{(p)} = \frac{1}{2\mu}(x_i \partial_j p + p\delta_{ij}) \tag{9.15}$$

Upon taking the divergence of the above equation, we get

$$\begin{aligned} \partial_j^2 u_i^{(p)} &= \frac{1}{2\mu}(2\delta_{ij}\partial_j p + x_i \partial_j^2 p) \\ &= \frac{1}{\mu}\partial_i p \end{aligned} \tag{9.16}$$

Therefore, the velocity fields can be obtained by solving the Laplace equation for the pressure and the homogeneous part of the velocity fields, and using equation 9.16 to obtain the particular part of the velocity field.

We have already studied the solution for the Laplace equation,

$$\nabla^2 \phi = 0 \tag{9.17}$$

which are obtained using separation of variables. In the present chapter, we derive the same solutions using the more convenient indicial notation. The fundamental solution for the Laplace equation in three dimensions, correct to within a multiplicative constant (which is determined from the boundary conditions, is

$$\Phi^{(0)} = \frac{1}{r} \tag{9.18}$$

A series of solutions can be derived from equation 9.18 as follows. If we take the gradient of equation 9.18, we get

$$\frac{\partial}{\partial x_i}\nabla^2\phi = \nabla^2\frac{\partial\phi}{\partial x_i} = 0 \tag{9.19}$$

Therefore, if ϕ is a solution of the Laplace equation, then its gradient $(\partial\phi/\partial x_i)$ is also a solution of the Laplace equation. The gradient of the fundamental solution is given by

$$\begin{aligned}\frac{\partial\Phi^{(0)}}{\partial x_i} & \\ &= \left(\mathbf{e}_1\frac{\partial}{\partial x_1} + \mathbf{e}_2\frac{\partial}{\partial x_2} + \mathbf{e}_3\frac{\partial}{\partial x_3}\right)\frac{1}{(x_1^2+x_2^2+x_3^2)^{1/2}} \\ &= -\frac{\mathbf{e}_1 x_1 + \mathbf{e}_2 x_2 + \mathbf{e}_3 x_3}{(x_1^2+x_2^2+x_3^2)^{3/2}} \\ &= -\frac{x_i}{r^3}\end{aligned} \tag{9.20}$$

Therefore, the second solution for the Laplace equation, which is proportional to the gradient of the first, is

$$\Phi_i^{(1)} = \frac{x_i}{r^3} \tag{9.21}$$

The third solution can be obtained by taking the gradient of the second solution,

$$\begin{aligned}\Phi_{ij}^{(2)} &= \frac{\partial\Phi_i^{(1)}}{\partial x_j} \\ &= \frac{\partial}{\partial x_j}\left(\frac{x_i}{r}\right) \\ &= \left(\frac{\delta_{ij}}{r} - \frac{3x_i x_j}{r^3}\right)\end{aligned} \tag{9.22}$$

It can easily be verified that the above solutions are linear combinations of the spherical harmonic solutions obtained by separation of variables. For example, the three components of $\Phi_i^{(1)}$ are

$$\Phi_3^{(1)} = \frac{x_3}{r^3}$$

$$
\begin{aligned}
&= \frac{\cos(\theta)}{r^2} \\
&= r^{-2}P_1^0(\cos(\theta)) \\
\Phi_1^{(1)} &= \frac{x_1}{r^3} \\
&= \frac{\sin(\theta)\cos(\phi)}{r^2} \\
&= r^{-2}P_1^1(\cos(\theta))((\exp(\imath\phi) + \exp(-\imath\phi))/2 \\
\Phi_2^{(1)} &= \frac{x_2}{r^3} \\
&= \frac{\sin(\theta)\sin(\phi)}{r^2} \\
&= r^{-2}P_1^1(\cos(\theta))((\exp(\imath\phi) - \exp(-\imath\phi))/(2\imath) \qquad (9.23)
\end{aligned}
$$

Similarly, it can be verified that the elements of the second solution $\Phi_{ij}^{(2)}$ are linear combinations of the second solution obtained from spherical harmonics, $(1/r^3)P_3^m(\cos(\theta))\exp(\imath m\phi)$. First note that the tensor $\Phi_{ij}^{(2)}$ is a symmetric tensor traceless tensor, and the sum of the diagonal elements is identically equal to zero. Therefore, there are five independent elements in this tensor, which correspond to the five solutions of the second spherical harmonic for $m = 0, \pm 1, \pm 2$. The component $\Phi_{33}^{(2)}$ is

$$
\begin{aligned}
\Phi_{33}^{(2)} &= \left(\frac{1}{r^3} - \frac{3x_3^2}{r^5}\right) \\
&= \frac{1}{r^3}(1 - 3\cos(\theta)^2) \\
&= -\frac{2P_3^0(\cos(\theta))}{r^3} \qquad (9.24)
\end{aligned}
$$

Similarly, the correspondence between the other elements of $\Phi_{ij}^{(2)}$ and the other spherical harmonic solutions can be shown.

The growing spherical harmonic solutions are obtained by multiplying the decaying solutions by r^{2n+1}, to obtain the series

$$
\begin{aligned}
\Phi^{(0)} &= 1 \\
\Phi^{(1)} &= x_i \\
\Phi^{(2)} &= r^2\delta_{ij} + 3x_ix_j \qquad (9.25)
\end{aligned}
$$

The solutions for the homogeneous part of the velocity field and the pressure field are linear combinations of the growing and decaying solutions.

9.1.1 Force on a settling sphere

In this section, we use the spherical harmonic expansion in order to determine for force exerted on a sphere settling in a fluid. The sphere is considered to be of unit radius, so that all length scales in the problem are scaled by the radius of the sphere. The sphere has a velocity $\mathbf{U}$, and the boundary conditions at the surface of the sphere are that the fluid velocity is equal to the velocity of the sphere,

$$u_i = U_i \text{ at } r = 1 \tag{9.26}$$

while the velocity decreases to zero at a large distance from the center of the sphere.

We use a spherical coordinate system, where the origin of the coordinate system is fixed at the center of the sphere, as shown in figure 9.3. Since the Stokes equations are linear, the fluid velocity and pressure fields are linear functions of the velocity U_i of the sphere. In addition, the homogeneous solution for the velocity field and the pressure are solutions of the Laplace equation, and so these are in the form of spherical harmonics given in 9.18, 9.20, 9.22 and the higher harmonics. The only linear combinations of the velocity of the sphere U_i and the spherical harmonic solutions which can result in a velocity vector are

$$u_i^{(h)} = A_1 U_i \frac{1}{r} + A_2 U_j \left(\frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5} \right) \tag{9.27}$$

where A_1 and A_2 are constants. Similarly, the only linear combinations of the velocity vector and the spherical harmonic solutions which can result in the scalar pressure is

$$p = A_3 U_j \frac{x_j}{r^3} \tag{9.28}$$

where A_3 is a constant. The total velocity field is the sum of the homogeneous and particular parts,

$$u_i = A_1 U_i \frac{1}{r} + A_2 U_j \left(\frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5} \right) + \frac{A_3 x_i U_j}{2\mu} \frac{x_j}{r^3} \tag{9.29}$$

The values of the constants A_1, A_2 and A_3 are determined from the boundary conditions at the surface of the sphere, 9.26, as well as the incompressibility condition 9.1 which has not been enforced yet. The incompressibility

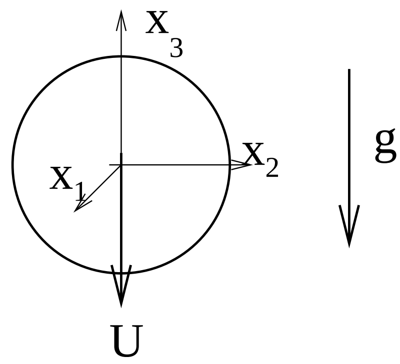


Figure 9.3: Sphere settling in a fluid.

condition $(\partial u_i/\partial x_i) = 0$ can be simplified as

$$\begin{aligned}
\frac{\partial u_i}{\partial x_i} &= A_1 U_i \frac{\partial}{\partial x_i}\frac{1}{r} + A_2 U_j \left(\frac{\partial}{\partial x_i}\frac{\delta_{ij}}{r^3} - 3\frac{\partial}{\partial x_i}\frac{x_i x_j}{r^5} \right) + \frac{A_3 U_j}{2\mu}\frac{\partial}{\partial x_i}\frac{x_i x_j}{r^3} \\
&= -\frac{A_1 U_i x_i}{r^3} + A_2 U_j \left(-\frac{3\delta_{ij}x_i}{r^5} - \frac{3\delta_{ij}x_i}{r^5} - \frac{3\delta_{ii}x_j}{r^5} + \frac{15 x_i^2 x_j}{r^7} \right) \\
&\quad + \frac{A_3 U_j}{2\mu}\left(\frac{\delta_{ii}x_j}{r^3} + \frac{\delta_{ij}x_i}{r^3} - \frac{3x_i^2 x_j}{r^5} \right) \\
&= -\frac{A_1 U_i x_i}{r^3} + \frac{A_3 U_j x_j}{2\mu r^3} \qquad (9.30)
\end{aligned}$$

In simplifying equation 9.30, we have used $x_i^2 = (x_1^2 + x_2^2 + x_3^2) = r^2$. Therefore, from the incompressibility condition, we get $A_1 = (A_3/2\mu)$. Inserting this into equation 9.30, the velocity field is

$$u_i = A_1 U_j \left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3} \right) + A_2 U_j \left(\frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5} \right) \qquad (9.31)$$

The constants A_1 and A_2 are determined from the boundary condition at the surface of the sphere $(r = 1)$,

$$\begin{aligned}
u_i|_{r=1} &= A_1 U_i + A_1 U_j x_i x_j + A_2 U_i - 3 A_2 U_j x_i x_j \\
&= U_i \qquad (9.32)
\end{aligned}$$

From this, we get two equations for A_1 and A_2,

$$\begin{aligned}
A_1 + A_2 &= 1 \\
A_1 - 3A_2 &= 0 \qquad (9.33)
\end{aligned}$$

These can be solved to obtain

$$A_1 = \frac{3}{4} \qquad A_2 = \frac{1}{4} \tag{9.34}$$

The final solution for the velocity field is

$$u_i = \frac{3U_j}{4}\left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right) + \frac{U_j}{4}\left(\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right) \tag{9.35}$$

while the pressure is

$$p = \frac{3\mu U_j x_j}{2r^3} \tag{9.36}$$

The force exerted on the sphere is given by

$$\begin{aligned} F_i &= \int_S dS T_{ik} n_k \\ &= \int dS(-p n_i + \tau_{ik} n_k) \end{aligned} \tag{9.37}$$

The shear stress is $\tau_{ik} = \mu((\partial u_i/\partial x_k) + (\partial u_k/\partial x_k))$. The gradient of the velocity field, $(\partial u_i/\partial x_k)$, can be determined as follows,

$$\begin{aligned} \frac{\partial u_i}{\partial x_k} &= \frac{3U_j}{4}\left(-\frac{\delta_{ij}x_k}{r^3} + \frac{\delta_{ik}x_j}{r^3} + \frac{\delta_{jk}x_i}{r^3} - \frac{3x_i x_j x_k}{r^5}\right) \\ &+ \frac{U_j}{4}\left(-\frac{3\delta_{ij}x_k}{r^5} - \frac{3\delta_{ik}x_j}{r^5} - \frac{3\delta_{jk}x_i}{r^5} + \frac{15 x_i x_j x_k}{r^7}\right) \end{aligned} \tag{9.38}$$

The gradient $(\partial u_k/\partial x_i)$ is obtained by interchanging the indices i and k in equation 9.38,

$$\begin{aligned} \frac{\partial u_k}{\partial x_i} &= \frac{3U_j}{4}\left(-\frac{\delta_{kj}x_i}{r^3} + \frac{\delta_{ik}x_j}{r^3} + \frac{\delta_{ij}x_k}{r^3} - \frac{3x_i x_j x_k}{r^5}\right) \\ &+ \frac{U_j}{4}\left(-\frac{3\delta_{jk}x_i}{r^5} - \frac{3\delta_{ik}x_j}{r^5} - \frac{3\delta_{ij}x_k}{r^5} + \frac{15 x_i x_j x_k}{r^7}\right) \end{aligned} \tag{9.39}$$

Adding these two contributions and multiplying by the viscosity, and setting the radius equal to 1 at the surface of the sphere, we get

$$\begin{aligned} \tau_{ik} &= \mu\left(\frac{3U_j}{2}(\delta_{ik}x_j - 3x_i x_j x_k)\right. \\ &\left. + \frac{U_j}{2}(-3\delta_{ij}x_k - 3\delta_{ik}x_j - 3\delta_{jk}x_i + 15 x_i x_j x_k)\right) \end{aligned} \tag{9.40}$$

The unit normal to the surface of the sphere, n_i is also equal to the ratio of the displacement vector and the radius, $n_i = (x_i/r)$. Since the radius is 1 on the surface of the sphere, $n_i = x_i$. Therefore, the dot product of the stress tensor and the unit normal is

$$\begin{aligned}\tau_{ik}n_k &= \mu\Big(\frac{3U_j}{2}(x_i x_j - 3x_i x_j x_k^2) \\ &\quad + \frac{U_j}{2}(-3\delta_{ij}x_k^2 - 6x_i x_j + 15x_i x_j x_k^2)\Big) \\ &= \mu\left(-\frac{3U_i}{2} + \frac{3U_j x_i x_j}{2}\right)\end{aligned} \quad (9.41)$$

The contribution to the force on the sphere due to the pressure is given by

$$-pn_i|_{r=1} = -\frac{3\mu U_j x_i x_j}{2} \quad (9.42)$$

Adding up these two contributions, we get

$$T_{ij}n_j = -\frac{3U_i}{2} \quad (9.43)$$

Since the right side of 9.43 is independent of position on the surface of the sphere, the total force is the product of $T_{ij}n_j$ and the surface area of the sphere, which is 4π for a sphere of unit radius.

$$F_i = 6\pi\mu U_i \quad (9.44)$$

For a sphere of radius R, the dependence of the force on the radius is deduced on the basis of dimensional analysis,

$$F_i = 6\pi\mu R U_i \quad (9.45)$$

9.1.2 Effective viscosity of a suspension

A shear flow is applied to a suspension which consists of solid particles with radius R and volume fraction ϕ suspended in a fluid of viscosity μ, as shown in figure 9.4(a). We will assume, for simplicity, that the suspension is dilute, so that the disturbance to the flow due to one particle does not affect the surrounding particles. We would like to derive an average constitutive relation for the suspension of the form

$$\langle T_{ij}\rangle = -\langle p\rangle\delta_{ij} + 2\mu_{eff}\langle E_{ij}\rangle \quad (9.46)$$

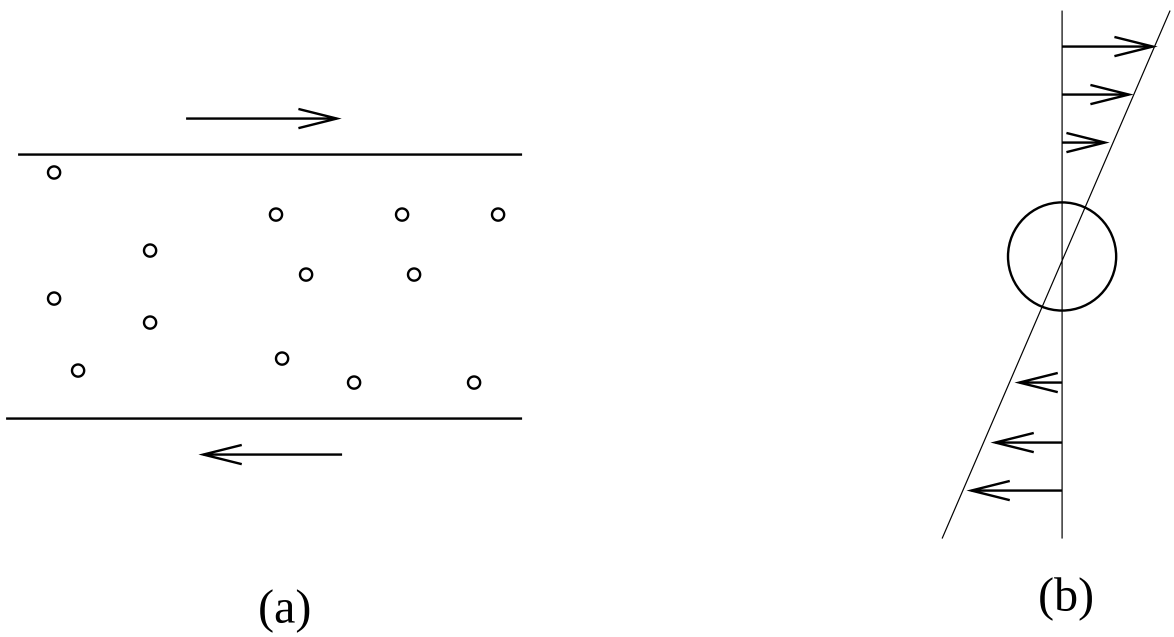


Figure 9.4: Shear flow of a suspension (a) and the shear flow around a particle (b).

where μ_{eff} is the effective viscosity of the suspension, and $\langle\rangle$ is a volume average over the entire volume of the suspension.

The volume averaged symmetric traceless part of the stress tensor can be expressed as

$$\begin{aligned}
\langle \tau_{ij} \rangle &= \frac{1}{V} \int_{V_{\text{suspension}}} dV \tau_{ij} \\
&= \frac{1}{V} \left(\int_{V_{\text{suspension}}} dV (2\mu E_{ij}) + \int_{V_{\text{suspension}}} dV (\tau_{ij} - 2\mu E_{ij}) \right) \\
&= 2\mu \langle E_{ij} \rangle + \frac{1}{V} \int_{V_{\text{suspension}}} dV (\tau_{ij} - 2\mu E_{ij}) \qquad (9.47)
\end{aligned}$$

The second term on the right side of 9.47 is identically equal to zero for the fluid, from the constitutive equation. Therefore, the second integral reduces to an integral over the particles,

$$\langle \tau_{ij} \rangle = 2\mu \langle E_{ij} \rangle + \frac{1}{V} \int_{V_{\text{particles}}} dV (\tau_{ij} - 2\mu E_{ij}) \qquad (9.48)$$

In order to determine the effective viscosity, it is sufficient to consider the symmetric traceless part of the above equation. Further, if the particles are solid, the rate of deformation E_{ij} within the particles is identically equal to zero. Therefore, the symmetric traceless part of the above equation reduces to

$$\begin{aligned}
\langle \tau_{ij} \rangle &= 2\mu \langle E_{ij} \rangle + \frac{N}{V} \int_{V_{1 \text{ particles}}} dV (\tau_{ij} - 2\mu E_{ij}) \qquad (9.49) \\
&= 2\mu \langle E_{ij} \rangle + \frac{N}{V} \int_{V_{1 \text{ particle}}} dV \tau_{ij} \qquad (9.50) \\
&= 2\mu \langle E_{ij} \rangle + \frac{\phi}{(4\pi R^3/3)} \int_{V_{1 \text{ particle}}} dV \tau_{ij} \qquad (9.51)
\end{aligned}$$

In deriving equation 9.51 from 9.50, we have used the simplification that the number of particles per unit volume is the ratio of the volume fraction and the volume of a particle. The second term on the right side of equation 9.51 can be simplified as follows. Consider the divergence of $(\tau_{il} x_j)$,

$$\frac{\partial \tau_{il} x_j}{\partial x_l} = \tau_{il} \delta_{lj} + x_j \frac{\partial \tau_{il}}{\partial x_l} \qquad (9.52)$$

$$= \tau_{ij} + x_j \frac{\partial \tau_{il}}{\partial x_l} \tag{9.53}$$

$$= \tau_{ij} \tag{9.54}$$

The divergence of the stress tensor, in the right side of equation 9.53, is zero if inertial effects are neglected, and therefore, the integral on the right side of equation 9.51 can be written as

$$\begin{aligned} \int_{V_{1 \text{ particle}}} dV \tau_{ij} &= \int_{V_{1 \text{ particle}}} dV \frac{\partial \tau_{il} x_j}{\partial x_l} \\ &= \int_{S_{1 \text{ particle}}} dS n_l \tau_{il} x_j \end{aligned} \tag{9.55}$$

The surface integral on the right side of equation 9.55 can be calculated using the spherical harmonic expansion, considering a sphere placed in a linear shear flow in which the fluid velocity at a large distance from the sphere, $u_i^\infty = E_{ij} x_j$. The fluid velocity field at a large distance from the sphere is given by

$$u_i = E_{ij} x_j \left(1 - \frac{R^5}{r^5}\right) + \frac{5 x_i x_j x_k E_{jk}}{2} \left(\frac{R^5}{r^7} - \frac{R^3}{r^5}\right) \tag{9.56}$$

and the pressure is

$$p = \frac{5 \mu R^3 x_j x_k E_{jk}}{r^5} \tag{9.57}$$

Using this, the surface integral on the left side of equation 9.55 can be calculated,

$$\int_{S_{1 \text{ particle}}} dS n_l \tau_{il} x_j = \frac{20 \pi R^3 \mu E_{ij}}{3} \tag{9.58}$$

With this, we get the effective shear stress

$$\begin{aligned} \langle \tau_{ij} \rangle &= 2 \mu E_{ij} + 5 \phi \mu E_{ij} \\ &= 2 \mu \left(1 + \frac{5\phi}{2}\right) E_{ij} \end{aligned} \tag{9.59}$$

Therefore, the effective viscosity is given by $\mu_{eff} = \mu(1 + 5\phi/2)$.

Exercise: Show that the velocity and pressure fields due to a sphere of radius R rotating with angular velocity $\boldsymbol{\Omega}$ in a fluid which is at rest a large distance from the sphere is,

$$u_i = \epsilon_{ijk}\Omega_j x_k \left(\frac{R}{r}\right)^3 \quad p = 0 \tag{9.60}$$

Show that the exerted by the sphere on the fluid is,

$$L_i = -8\pi\mu R^3 \Omega_i \tag{9.61}$$

Hint: Note that the angular velocity is a pseudo-vector, and the velocity field and pressure are real.

9.2 Green's functions and Faxen's laws:

9.2.1 Oseen tensor for a point force:

Since the Stokes equations are linear equations, it is possible to obtain Green's function formulations for the velocity field similar to those obtained for the temperature field in steady state heat conduction. We briefly revisit the solution for the conduction equation in chapter , and then we present the analogous solution for the Stokes equations. In chapter , we solved the conduction equation,

$$-k\nabla^2 T = Q\delta(\mathbf{r} - \mathbf{r}_s) \tag{9.62}$$

to obtain the temperature field,

$$T(\mathbf{r}) = \frac{Q}{4\pi K|\mathbf{r} - \mathbf{r}_s|} \tag{9.63}$$

where $\mathbf{r}_s$ is the location of the source point. In order to obtain the Green's function solution, we first obtained the temperature field for a sphere of radius R, and then took the limit $R \to 0$ while keeping the heat flux Q finite.

In viscous flows, the equations equivalent of the Green's function formulation 9.87 are,

$$\partial_i u_i = 0 \tag{9.64}$$

$$-\partial_i p + \mu\partial_j^2 u_i = F_i \delta(\mathbf{x} - \mathbf{x}_s) \tag{9.65}$$

where $\mathbf{x}_s$, the location of the force, is the 'source point'. Note that the right side of equation 9.90 has dimensions of force per unit volume, since F_i is the total force exerted at the point $\mathbf{x}_s$, and the delta function has units of inverse volume.

In order to solve equation 9.89 and 9.90, we can first solve for the velocity and pressure fields around a spherical particle of finite radius R, and then take the limit $R \to 0$ while keeping the force F_i finite. The solutions for the velocity and pressure fields were obtained in equations 9.35 and 9.36, and the force was obtained as a function of the translational velocity of the sphere in equation 9.44. If we use 9.44 to express the translational velocity in terms of the force in equations 9.35 and 9.36, we obtain,

$$u_i = \frac{F_j}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right) + \frac{F_j R^2}{24\pi}\left(\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right) \tag{9.66}$$

while the pressure is

$$p = \frac{F_j x_j}{4\pi r^3} \tag{9.67}$$

When we take the limit $R \to 0$, the first term on the right in equation 9.91 remains finite while the second becomes negligible, while the term on the right side of equation 9.92 is finite. Therefore, the velocity and pressure fields at a field point $\mathbf{x}$ due to a point force F_i exerted at the source point $\mathbf{x}^{(s)}$ is,

$$\begin{aligned} u_i &= J_{ij}(\mathbf{x} - \mathbf{x}_s) F_j(\mathbf{x}^{(s)}) \\ p &= K_i(\mathbf{x} - \mathbf{x}^{(s)}) F_i(\mathbf{x}^{(s)}) \end{aligned} \tag{9.68}$$

where the 'Oseen tensors', J_{ij} and K_i, are given by,

$$\begin{aligned} J_{ij}(\mathbf{x}) &= \frac{1}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right) \\ K_i(\mathbf{x}) &= \frac{1}{4\pi}\frac{x_i}{r^3} \end{aligned} \tag{9.69}$$

where $r = |\mathbf{x}|$ is the distance from the source point to the field point.

The linearity of the Stokes equations enables us to use superposition to find the velocity field due to multiple forces. For example, if the are N point forces $F_i^{(n)}$ located at positions $\mathbf{x}^{(n)}$, where the index n varies from 1 to N, then the total velocity field is just the superposition of the velocity fields due to each of these forces,

$$\begin{aligned} u_i(\mathbf{x}) &= \sum_{n=1}^{N} J_{ij}(\mathbf{x} - \mathbf{x}^{(n)}) F_j^{(n)}(\mathbf{x}^{(n)}) \\ p(\mathbf{x}) &= K_i(\mathbf{x} - \mathbf{x}^{(n)}) F_i(\mathbf{x}^{(n)}) \end{aligned} \tag{9.70}$$

Instead of having a discrete set of point forces, if we have a force distribution, in which the force per unit volume is given by $f_i(\mathbf{x})$, the velocity and pressure fields can be calculated by integrating the appropriate product of the Oseen tensor and the force distribution over the volume,

$$\begin{aligned} u_i(\mathbf{x}) &= \int d\mathbf{x}' J_{ij}(\mathbf{x}-\mathbf{x}') f_j(\mathbf{x}') \\ p(\mathbf{x}) &= \int d\mathbf{x}' K_i(\mathbf{x}-\mathbf{x}') F_i(\mathbf{x}') \end{aligned} \tag{9.71}$$

9.2.2 Green's function for force dipoles:

There are often situations where the suspended particles in a fluid a 'neutrally buoyant', so that they exert no net force on the fluid. Examples are for a solid sphere in a uniform shear flow in equations **??** and **??**, and for a rotating particle in equations **??** and **??**. The velocity disturbance at a large distance from the center of the particle decays proportional to r^{-2}, in contrast to the decay proportional to r^{-1} in equation **??** for the flow around a particle which exerts a net force on the fluid. The far-field velocity can be expressed as a function of the 'force dipole moment', which is defined as the integral over the sphere of the tensor product of the surface force and the position vector,

$$\begin{aligned} \text{Force dipole moment} &= \int dS x_i f_j \\ &= \int dS x_i \tau_{jk} n_k \end{aligned} \tag{9.72}$$

Note that this definition is analogous to the 'dipole moment' for the temperature field due to a point dipole. The force dipole moment is a second order tensor, and it is convenient to separate it into a symmetric and an antisymmetric part. The symmetric part is given by,

$$S_{ij} = \frac{1}{2} \int dS (x_i \tau_{jk} n_k + x_j \tau_{ik} n_k) \tag{9.73}$$

while the antisymmetric part is,

$$A_{ij} = \frac{1}{2} \int dS (\tau_{ik} n_k x_j - \tau_{jk} n_k x_i) \tag{9.74}$$

The velocity and pressure fields around a point particle which exerts no net force on the fluid can be expressed in terms of the symmetric and antisymmetric dipole moments S_{ij} and A_{ij}.

In the previous sub-section, we had derived the velocity field due to a point force by first solving the flow around a sphere of finite radius R, and then taking the limit $R \to 0$ while keeping the force exerted a constant. The velocity field due to a force dipole can be obtained by first solving for the shear flow around a particle for radius R, and taking the limit of the radius going to zero while keeping the dipole moment fixed. First, we consider the symmetric dipole moment due to a sphere in a shear flow, for which the velocity and pressure fields are given by equations **??** and **??**. It can easily be shown, using symmetry considerations, that the net force exerted by the sphere on the fluid is identically zero. The symmetric force dipole due to the sphere is given by,

$$\begin{aligned} S_{ij} &= \frac{1}{2}\int_S dS(f_i x_j + x_i f_j) \\ &= \frac{1}{2}\int_S dS(\tau_{il} n_l x_j + \tau_{jl} n_l x_i) \end{aligned} \tag{9.75}$$

The integrals on the right side of equation **??** were already evaluated in equation **??**, and the final result for the symmetric force dipole is,

$$S_{ij} = \frac{20\pi\mu R^3 E_{ij}}{3} \tag{9.76}$$

The rate of deformation tensor E_{ij} can be written in terms of the symmetric force dipole using equation **??** and substituted into equations **??** and **??** for the velocity and pressure fields.

$$u_i = E_{ij}x_j - \frac{3S_{ij}x_j R^2}{20\pi\mu r^5} + \frac{3x_i x_j x_k S_{jk}}{8\pi\mu}\left(\frac{R^5}{r^7} - \frac{R^3}{r^5}\right) \tag{9.77}$$

and the pressure is

$$p = \frac{3x_j x_k S_{jk}}{4r^5} \tag{9.78}$$

In equation **??**, the first term on the right side is the imposed flow and not the disturbance due to the particle, and so we have not expressed this in terms of S_{ij}. If we take the point particle limit $R \to 0$ while keeping S_{ij} a constant, we obtain the solution for the velocity field due to a symmetric point dipole,

$$u_i = E_{ij}x_j - \frac{3S_{jk}x_i x_j x_k}{8\pi\mu r^5} \tag{9.79}$$

while the pressure field is given by equation **??**.

The response to an antisymmetric force dipole can be related to the torque exerted by the particle on the fluid, which is defined as,

$$\begin{aligned} L_i &= \int_S dS \epsilon_{ijk} x_j f_k \\ &= \frac{1}{2} \int_S dS \epsilon_{ijk} (x_j f_k - x_k f_j) \\ &= \epsilon_{ijk} D^A_{jk} \end{aligned} \tag{9.80}$$

where D^A_{jk} is the antisymmetric force dipole moment exerted by the particle. The torque is related to the angular velocity by equation **??**, while the pressure is zero. Substituting for the particle velocity in terms of the torque, we obtain,

$$u_i = -\frac{\epsilon_{ijk} L_j}{8\pi\mu} \left(\frac{x_k}{r^3} \right) \quad p = 0 \tag{9.81}$$

9.2.3 Faxen's laws

Faxen's laws provide the velocity of a particle which is placed in some imposed flow field $\mathbf{U}^\infty(\mathbf{x})$. We have studied examples where the imposed flow field has a specific form; in section **??**, the imposed flow field is a constant streaming flow around the particle, while in section **??**, the imposed flow field is a linear shear flow. In this section, we examine the response of a spherical particle to an arbitrary imposed flow field, $\mathbf{U}^\infty(\mathbf{x})$, which is the velocity field at a large distance from the particle. This is a first step towards understanding inter-particle interactions, because if we know how the particle reacts to some imposed flow field, we can infer how it would react to the velocity disturbance caused by a second particle.

The fluid velocity field due to a particle in the flow field $\mathbf{U}^\infty(\mathbf{x})$ can be expressed in terms of the surface forces using the Oseen tensor,

$$\begin{aligned} u_i(\mathbf{x}) &= U_i^\infty(\mathbf{x}) + \int_S' dS' J_{ij}(\mathbf{x} - \mathbf{x}') f_j(\mathbf{x}') \\ &= U_i^\infty(\mathbf{x}) + \int_S' dS' J_{ij}(\mathbf{x} - \mathbf{x}') \tau_{jk}(\mathbf{x}') n'_k \end{aligned} \tag{9.82}$$

where $\mathbf{x}'$ is a point on the surface of the particle, S' is the surface of the particle and $\mathbf{n}'$ is the unit normal to the surface at the position $\mathbf{x}'$. Equation **??** also applies to the fluid velocity at the surface of the particle. The fluid velocity at a location on the surface, $\mathbf{x}''$, which is equal to the particle velocity at that location $\mathbf{U} + \boldsymbol{\Omega} \times \mathbf{x}''$, can be written as,

$$\begin{aligned} u_i(\mathbf{x}'') &= U_i + \epsilon_{ijk}\Omega_j x''_k \\ U_i^{\infty}(\mathbf{x}'') + \int_S' dS' J_{ij}(\mathbf{x}'' - \mathbf{x}')\tau_{jk}(\mathbf{x}')n'_k & \end{aligned} \tag{9.83}$$

Here $\mathbf{U}$ is the translational velocity of the particle, and $\boldsymbol{\Omega}$ is the rotational velocity. Note that in equation **??**, both the field point $\mathbf{x}''$ and the source point $\mathbf{x}'$ are located on the surface of the sphere. Now, we integrate the equation **??** over the surface of the sphere, and use the no-slip condition at the surface which states that the fluid velocity $\mathbf{u}$.

$$4\pi R^2 U_i = \int_S dS'' U_i^{\infty}(\mathbf{x}'') + \int_S dS'' \int_S dS' J_{ij}(\mathbf{x}'' - \mathbf{x}')\tau_{jk}(\mathbf{x}')n'_k \tag{9.84}$$

Note that the integral of the velocity due to partice rotation at the surface, $\int_S dS''\epsilon_{ijk}\Omega_j x''_k$ is zero, since the integrand is an odd function of the position vector on the surface. The second term on the right side of equation **??** can be simplified by changing the order of integration,

$$\int_S dS'' \int_S dS' J_{ij}(\mathbf{x}'' - \mathbf{x}')\tau_{jk}(\mathbf{x}')n'_k = \int_S dS' \tau_{jk}(\mathbf{x}')n'_k \int_S dS'' J_{ij}(\mathbf{x}'' - \mathbf{x}') \tag{9.85}$$

The integral over S'' in equation **??**, which is a constant independent of $\mathbf{x}'$, was shown to be equal to $(2\delta_{ij}R/(3\pi\mu))$ in **??**, and the integal $\int_S dS'\tau_{jk}(\mathbf{x}')n'_k$ is just the total force exerted by the sphere, F_j. With this, equation **??** simplifies to

$$U_i = \frac{F_i}{6\pi\mu R} + \int_S dS'' U_i^{\infty}(\mathbf{x}'') \tag{9.86}$$

This is the integral form of the Faxen law for the velocity of a particle in an imposed velocity field $\mathbf{U}^{\infty}$.

9.3 Lubrication flow

The flow of fluids through thin layers between solid surfaces are referred to as 'lubrication flows'. These flows are dominated by viscous effects, since the Reynolds number based on the relevant length scale, which is the distance between the solid surfaces, is small. The approach of the two surfaces requires the fluid to be squeezed out of the gap between the surfaces, and the velocity of the fluid tangential to the surfaces is large compared to the velocity with which the surfaces approach each other. This results in large shear stresses in the gap, and a large resistance to the relative motion of the surfaces. This flow can be analysed using asymptotic analysis when the gap thickness is small. The lubrication flow generated due to a sphere approaching a solid surface is analysed in this section.

The configuration, shown in figure 9.5, consists of a sphere of radius R approaching a plane surface with a velocity U. The minimum distance between the surface of the sphere and the plane surface is ϵR, where $\epsilon \ll 1$. A cylindrical coordinate system is used, since there is cylindrical symmetry about the perpendicular to the surface that passes through the center of the sphere, and the origin is fixed at the intersection between the axis of symmetry and the plane surface. There is no dependence on the polar angle in this problem, and so the velocity field is a function of the distance from the axis r and the height above the plane surface z. The plane surface is located at $z = 0$, while the equation for the surface of the sphere is given by

$$((R(1+\epsilon) - z)^2 + r^2)^{1/2} = R \tag{9.87}$$

Since ϵ is small, the above equation can be expanded in a Taylor series in ϵ. The equation for the bottom surface is

$$R(1+\epsilon) - z = \sqrt{R^2 - r^2} \tag{9.88}$$

$$z = R(1+\epsilon) - \sqrt{R^2 - r^2} \tag{9.89}$$

We anticipate that r is also small compared to R in this region, and so the second term on the right side of equation 9.89 can be expanded in small r. When this expansion is carried out, and the first term in the series is retained, we get

$$\begin{aligned} z &= R(1+\epsilon) - R\left(1 - \frac{r^2}{2R^2}\right) \\ &= R\epsilon + \frac{r^2}{2R} \end{aligned} \tag{9.90}$$

Dividing throughout by $R\epsilon$, the equation for the surface is

$$\frac{z}{R\epsilon} = 1 + \frac{r^2}{2R^2\epsilon^2} \tag{9.91}$$

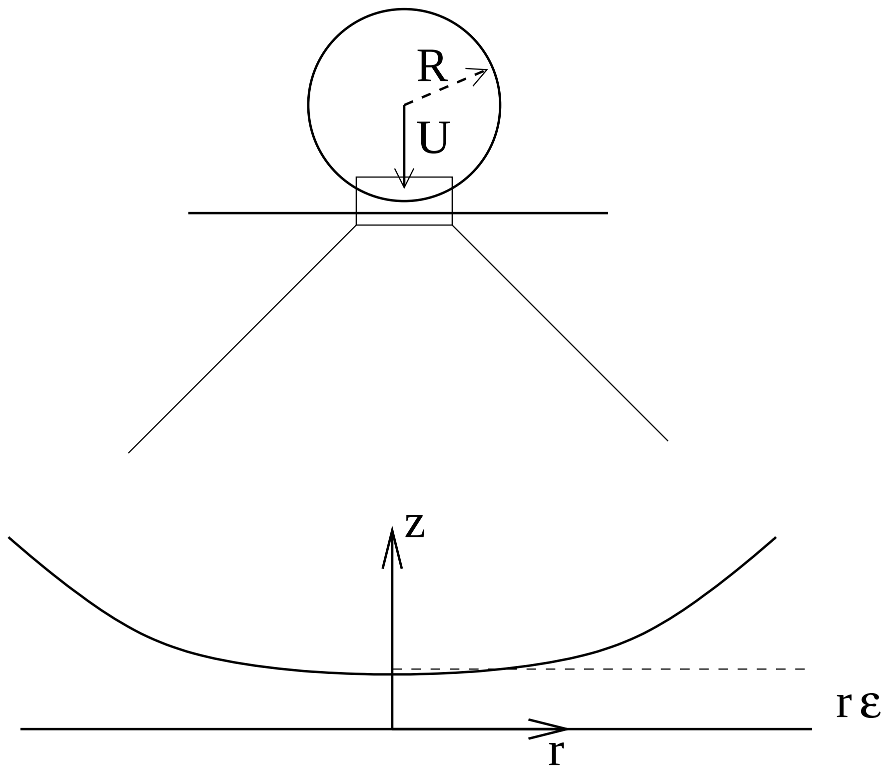


Figure 9.5: Lubrication flow in a small gap between a sphere and a solid surface.

This equation suggests the use of the scaled coordinates $z^* = (z/R\epsilon)$ and $r^* = (r/R\epsilon^{1/2})$. The equation for the surface in terms of these scaled coordinates is

$$z^* = h(r^*) = 1 + \frac{r^{*2}}{2} \tag{9.92}$$

This is called the 'parabolic approximation' for the surface of the sphere. The scaling $z^* = (z/R\epsilon)$ is natural, since the fluid is in the domain $0 < z < R\epsilon$ at $r = 0$. The scaling $r^* = (r/R\epsilon^{1/2})$ arises because the slope (dr/dz) of the surface of the sphere is zero at $r = 0$, and so the surface can be approximated by a parabola around this point.

Next, the components of the velocity in the fluid are scaled using the mass conservation equation.

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_z}{\partial z} = 0 \tag{9.93}$$

The axial velocity u_z can be scaled by the velocity of the surface of the sphere, U, since the axial velocity varies between 0 at the plane surface and $-U$ at the surface of the sphere. When the mass conservation equation is expressed in terms of the scaled coordinates and the scaled velocity $u_z^* = (u_z/U)$, we get

$$\frac{1}{R\epsilon^{1/2}}\frac{1}{r^*}\frac{\partial(r^* u_r)}{\partial r^*} + \frac{U}{\epsilon R}\frac{\partial u_z^*}{\partial z^*} = 0 \tag{9.94}$$

The above equation provides the scaled for the radial velocity is $u_r^* = (u_r \epsilon^{1/2}/U)$. This scaling indicates that when u_r^* is $O(1)$, the magnitude of the radial velocity is $u_r \sim (U/\epsilon^{1/2})$, which is large compared to the magnitude of the axial velocity. This is because as the sphere moves downward, all the fluid displaced per unit time within a cylindrical section of radius r is $\pi r^2 U$. This fluid has to be expelled from the cylindrical surface with area $2\pi r h$, and therefore the radial velocity scales as (Ur/h), which is proportional to $(U/\epsilon^{1/2})$.

The momentum conservation equations in the radial direction, expressed in terms of the scaled variables are

$$\frac{\rho U^2}{\epsilon^{3/2}}\left(\frac{\partial u_r^*}{\partial t^*} + u_r^*\frac{\partial u_r^*}{\partial r^*} + u_z^*\frac{\partial u_r^*}{\partial z^*}\right) = -\frac{1}{R\epsilon^{1/2}}\frac{\partial p}{\partial r^*} + \mu\left(\frac{U}{R^2\epsilon^{5/2}}\frac{\partial^2 u_r^*}{\partial z^{*2}} + \frac{U}{R^2\epsilon^{3/2}}\frac{1}{r^*}\frac{\partial}{\partial r^*}r^*\frac{\partial u_r^*}{\partial r^*}\right) \tag{9.95}$$

In the above equation, the scaled time has been defined as $t^* = (t\epsilon R/U)$, since the relevant time scale is the ratio of the gap width and the velocity of the sphere. This ensures that the first term on the left side of equation 9.95 is of the same magnitude as the other three terms. The momentum conservation can be simplified by dividing throughout by the coefficient of the largest viscous term, which is proportional to $(\mu U/R^2\epsilon^{5/2})$,

$$\frac{\rho U R\epsilon}{\mu}\left(\frac{\partial u_r^*}{\partial t^*} + u_r^*\frac{\partial u_r^*}{\partial r^*} + u_z^*\frac{\partial u_r^*}{\partial z^*}\right) = -\frac{R\epsilon^2}{\mu U}\frac{\partial p}{\partial r^*} + \left(\frac{\partial^2 u_r^*}{\partial z^{*2}} + \frac{\epsilon}{r^*}\frac{\partial}{\partial r^*}r^*\frac{\partial u_r^*}{\partial r^*}\right) \tag{9.96}$$

This equation indicates that the scaled pressure should be defined as $p^* = (pR\epsilon^2/\mu U)$ for the pressure to be of the same magnitude as the viscous terms. Equation 9.96 also indicates that the appropriate Reynolds number in this case is $(\rho U R\epsilon/\mu)$, which is based on the gap thickness and the velocity of the sphere. Inertial effects can be neglected when this Reynolds number is small, even though the Reynolds number based on the sphere velocity and the sphere radius and sphere velocity is large. After neglecting inertial effects, and retaining only the largest terms in the ϵ expansion, the momentum conservation equation in the radial direction becomes

$$-\frac{\partial p^*}{\partial r^*} + \frac{\partial^2 u_r^*}{\partial z^{*2}} = 0 \tag{9.97}$$

The momentum conservation equation in the axial direction, expressed in terms of the scaled coordinates, velocity and pressure, is

$$\frac{\rho U^2}{R\epsilon}\left(\frac{\partial u_z^*}{\partial t^*} + u_r^*\frac{\partial u_z^*}{\partial r^*} + u_z^*\frac{\partial u_z^*}{\partial z^*}\right) = -\frac{\mu U}{R\epsilon^3}\frac{\partial p^*}{\partial z^*} + \mu\left(\frac{U}{R^2\epsilon^2}\frac{\partial^2 u_z^*}{\partial z^{*2}} + \frac{U}{R^2\epsilon}\frac{1}{r^*}\frac{\partial}{\partial r^*}r^*\frac{\partial u_z^*}{\partial r^*}\right) \tag{9.98}$$

It is evident that the largest term in the equation is the pressure gradient, since it is multiplied by $(\mu U/R\epsilon^3)$. Dividing throughout by this factor, we get

$$\frac{\rho U R\epsilon^2}{\mu}\left(\frac{\partial u_z^*}{\partial t^*} + u_r^*\frac{\partial u_z^*}{\partial r^*} + u_z^*\frac{\partial u_z^*}{\partial z^*}\right) = -\frac{\partial p^*}{\partial z^*} + \epsilon\left(\frac{\partial^2 u_z^*}{\partial z^{*2}} + \frac{\epsilon}{r^*}\frac{\partial}{\partial r^*}r^*\frac{\partial u_z^*}{\partial r^*}\right) \tag{9.99}$$

If all terms that are small in the ϵ expansion are neglected, the axial momentum conservation equation becomes

$$\frac{\partial p^*}{\partial z^*} = 0 \tag{9.100}$$

Equation 9.100 indicates that the pressure is independent of the z^* coordinate. Using this, equation 9.97 can be solved to obtain the radial velocity,

$$u_r^* = \frac{\partial p^*}{\partial r^*}\frac{z^{*2}}{2} + A_1(r^*)z^* + A_2(r^*) \tag{9.101}$$

The functions $A_1(r^*)$ and $A_2(r^*)$ are obtained from the no-slip condition $u_r^* = 0$ at $z^* = 0$ and the surface of the sphere $z^* = h(r^*)$. The radial velocity then becomes

$$u_r^* = \frac{\partial p^*}{\partial r^*}\left(\frac{z^{*2}}{2} - \frac{z^* h(r^*)}{2}\right) \tag{9.102}$$

At this point, the radial pressure gradient is not yet specified, and we have not used the mass conservation equation so far. So the mass conservation equation can be used to obtain the radial pressure gradient. It is convenient to integrate the mass conservation equation over the width of the channel to get

$$\int_0^{h(r^*)} dz^* \frac{1}{r^*}\frac{\partial}{\partial r^*} r^* \frac{\partial u_r^*}{\partial r^*} + \int_0^{h(r^*)} dz^* \frac{\partial u_z^*}{\partial z^*} \;\; = \;\; 0 \tag{9.103}$$

This can be simplified to provide

$$\begin{aligned} \frac{1}{r^*}\frac{\partial}{\partial r^*} r^* \frac{\partial}{\partial r^*}\int_0^{h(r^*)} dz^* u_r^* - 1 &= 0 \\ \frac{1}{r^*}\frac{\partial}{\partial r^*} r^* \frac{\partial}{\partial r^*}\left(-\frac{h(r^*)^3}{12}\frac{\partial p^*}{\partial r^*}\right) - 1 &= 0 \end{aligned} \tag{9.104}$$

This equation can be integrated to provide the radial pressure gradient,

$$\frac{\partial p^*}{\partial r^*} = -\frac{6r^*}{h(r^*)^3} - \frac{C_1}{r^* h(r^*)^3} \tag{9.105}$$

The condition that the pressure is finite throughout the channel requires that the constant C_1 is zero. A second integration with respect to the radial coordinate provides the pressure.

$$p^* = \frac{3}{(1 + (r^{*2}/2))^2} + C_2 \tag{9.106}$$

The pressure C_2 is obtained from the condition that the dimensional pressure scales as $(\mu U/R)$ in the limit $r^* \to 0$, and therefore the scaled pressure is proportional to ϵ^3 in this limit.

$$p^* = \frac{3}{(1 + (r^{*2}/2))^2} \tag{9.107}$$

The scaled force in the z^* direction is obtained by integrating the axial component of the product of the stress tensor and the unit normal,

$$F_z = 2\pi \int r \, dr (T_{zz} n_z + T_{zr} n_r) \tag{9.108}$$

The unit normals to the surface can be estimated as

$$\begin{aligned} n_z &= \frac{1}{(1+(dh/dr)^2)^{1/2}} \\ &= \frac{1}{(1+\epsilon(dh^*/dr^*)^2)^{1/2}} \end{aligned} \tag{9.109}$$

$$\begin{aligned} n_r &= \frac{(dh/dr)}{(1+(dh/dr)^2)^{1/2}} \\ &= \frac{\epsilon^{1/2}(dh^*/dr^*)}{(1+\epsilon(dh^*/dr^*)^2)^{1/2}} \end{aligned} \tag{9.110}$$

From the above, it is observed that n_z is 1 in the leading approximation in small ϵ, while n_r scales as $\epsilon^{1/2}$. Therefore the leading contribution to 9.108 is due to the z component of the stress tensor,

$$\begin{aligned} F_z &= 2\pi \int r\, dr\, T_{zz} \\ &= 2\pi \int r\, dr \left(-p + 2\mu \frac{\partial u_r}{\partial r}\right) \end{aligned} \tag{9.111}$$

The above equation can further be simplified by realising that the pressure scales as $(\mu U/\epsilon^2 R)$, whereas the normal stress due to viscous effects $\mu(\partial u_r/\partial r) \sim (\mu U/\epsilon^{1/2} R)$. Therefore, the dominant contribution to $T_z z$ is due to the pressure,

$$\begin{aligned} F_z &= -2\pi \int r\, dr\, p \\ &= -\frac{2\pi\mu U R}{\epsilon} \int r^* dr^*\, p^* \\ &= -\frac{2\pi\mu U R}{\epsilon} \int r^* dr^* \frac{3}{(1+(r^{*2}/2))^2} \\ &= \frac{6\pi\mu U R}{\epsilon} \end{aligned} \tag{9.112}$$

9.4 Inertial effects at low Reynolds number:

We next examine the effect of fluid inertia in the low Reynolds number limit. The simplest problem one can consider is the settling of a sphere in the low

Reynolds number limit. The zero Reynolds number solution for this problem was obtained in section **??**, where it was shown that the velocity and pressure fields are,

$$u_i = \frac{3U_j}{4}\left(\frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3}\right) + \frac{U_j}{4}\left(\frac{\delta_{ij}}{r^3} - \frac{3x_i x_j}{r^5}\right) \tag{9.113}$$

while the pressure is

$$p = \frac{3\mu U_j x_j}{2r^3} \tag{9.114}$$

Problems:

1. Two spheres of equal radius are falling due to gravity in a viscous fluid, and the line joining their centres is at an angle θ to the vertical as shown in Figure 1. The velocities of the two spheres, $\mathbf{u}_1$ and $\mathbf{u}_2$, can be separated into the velocity of the centre of mass $\mathbf{v} = (\mathbf{u}_1 + \mathbf{u}_2)/2$ and the velocity difference between the spheres $\mathbf{w} = (\mathbf{u}_1 - \mathbf{u}_2)$. These can further be separated into the components along the line joining the centres, $\mathbf{v}_{\|}$ and $\mathbf{w}_{\|}$, and the velocities perpendicular to the line of centres, $\mathbf{v}_{\perp}$ and $\mathbf{w}_{\perp}$. Using Stokes flow reversibility and symmetry, determine which of these four components can be non - zero, and which are identically zero.

2. G. I. Taylor showed that the sedimentation velocity $\mathbf{U}_{\|}$ of a needle like object (or slender body) when its axis is vertical (parallel to gravity) is twice the sedimentation velocity $\mathbf{U}_{\perp}$ when the axis is perpendicular to gravity, i. e., $\mathbf{U}_{\|} = 2\mathbf{U}_{\perp}$. Using this information, find a general equation that relates the sedimentation velocity U_i and orientation p_i of the slender body to $U_{\|}$ and g_i, the unit vector in the direction of gravity. Assume zero Reynolds number flow.

3. A *two dimensional* system consists of an object having the shape of an *equilateral triangle* with three equal sides settling in a fluid as shown in Figure 1(a) in the limit of zero Reynolds number, where inertia is neglected.

   (a) Sketch the velocity profile you would expect for the flow around this object shown in figure 1(a) in which one side is perpendicular to the direction of gravity. From linearity and symmetry, would you expect the horizontal velocity to be zero? From linearity and symmetry, would you expect the rotational velocity of the object about its center of mass to be zero? Give reasons.

(b) If the object were to settle as shown in figure 1(b), with side parallel to the direction of gravity, sketch the velocity profile you would expect. From linearity and symmetry, would you expect the horizontal velocity to be zero? From linearity and symmetry, would you expect the rotational velocity of the object about its center of mass to be zero? Give reasons.

(c) If the object settles with all its sides neither parallel nor perpendicular to the direction of gravity as in figure 1(c), what is the magnitude and direction of the velocity, as a function of the velocities in figures 1(a) and 1(b)?

(d) If the fluid flow is described by the Stokes equations, what is the form of the velocity field due to this object when the distance from the object is large compared to the length of one side?

(e) What is the distance form the object upto which the inertial terms can be neglected?

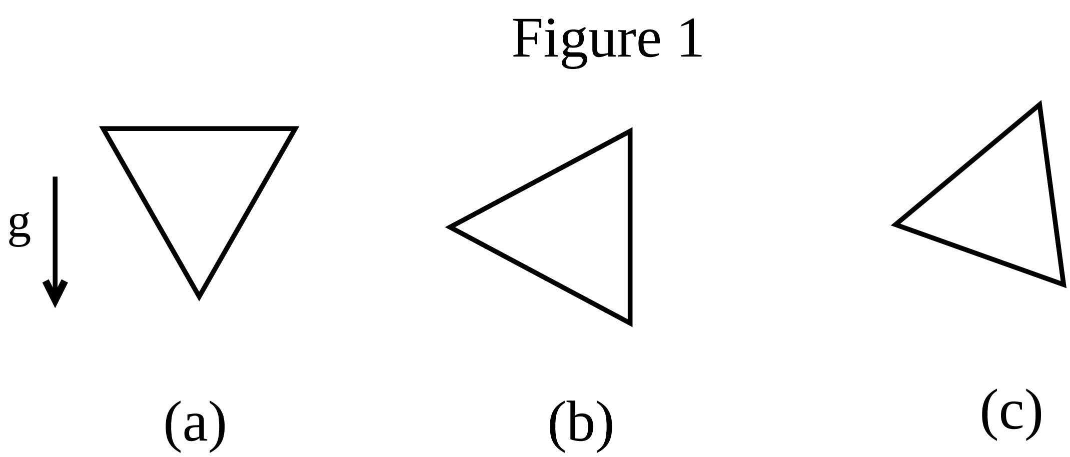


Figure 9.6:

4. Determine the solutions for the velocity field in two dimensions for a cylinder of infinite length moving perpendicular to its axis in the limit of zero Reynolds number, as follows.

(a) Write down the Stokes equations in terms of two Laplace equations, one for the velocity and one for the pressure fields.

(b) Solve the Laplace equations for a point source which is independent of the θ co-ordinate.

(c) Obtain the higher harmonics by taking gradients of the fundamental solution.

(d) What are the possible solutions for the velocity and pressure fields based on vector symmetries?

(e) Obtain the constants in the solutions from the boundary conditions.

5. Determine the fluid flow field and the stress acting on a particle of radius a placed in an extensional strain field $u_i = G_{ij} x_j$ ($G_{ij} = G_{ji}$) at low Reynolds number where the fluid and particle inertia can be neglected. The particle is placed at the origin of the coordinate system, and the fluid velocity field has the undisturbed value $u_i = G_{ij} x_j$ for $r \to \infty$. Find the fluid velocity and pressure fields around the particle. Find the integral:
$$\int_A dA T_{il} n_l x_j \tag{9.115}$$
over the surface of the sphere, where n_l is the outward unit normal.

6. A sphere is rotating with an angular velocity Ω_k in a fluid that is at rest at infinity, and the Reynolds number based on the angular velocity and radius of the sphere, $Re \equiv \rho \Omega a^2/\mu$ is small. The velocity at the surface of the particle is:
$$u_i|_{r=a} = \epsilon_{ikl} \Omega_k x_l \tag{9.116}$$
and the velocity and pressure fields decay far from the sphere.

(a) Find the most general form for the velocity and pressure fields at zero Reynolds number. Note that the velocity and pressure are real vectors, whereas Ω_k is a pseudo vector.

(b) Use the incompressibility condition and the condition on the velocity at the surface of the sphere to determine the constants in your expression for the fluid velocity.

(c) Determine the force per unit area f_i exerted on the surface of the sphere. Note that the force is a function of position on the surface of the sphere.

(d) The torque on the particle is

$$L_i = \int_A dA \epsilon_{ijk} f_j n_k \tag{9.117}$$

where f_j is the force per unit area exerted on the sphere at position x_n. Find the torque.

7. Find the force necessary to move a disk of radius a towards a plane solid boundary with a velocity U when the gap between the disk and the boundary h is small, $h = \epsilon a$ where $\epsilon \ll 1$.

8. A two dimensional cylinder of radius a is moving with a velocity U along the center of a two dimensional channel of width $2a(1+\epsilon)$, where $\epsilon \ll 1$ as shown in Figure 1. The end of the slot is closed so that the fluid displaced by the cylinder has to escape through the narrow gap between the cylinder and the channel.

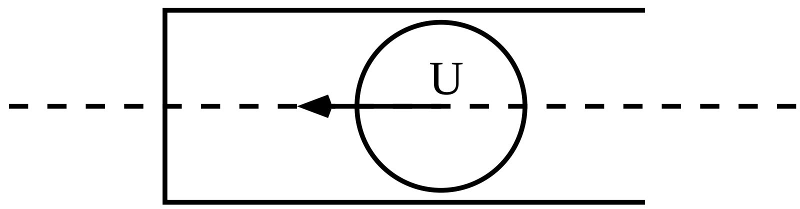
   (a) Choose a coordinate system for analysing the flow in the gap between the cylinder and the channel wall. Write the Navier Stokes equations for the flow in the gap, and scale the equations appropriately. Under what conditions can the inertial terms in the conservation equation be neglected?

   (b) What are the boundary conditions required to solve the problem? What is the other condition due to the requirement that the volume displaced by the cylinder has to flow through the gap?

   (c) Determine the velocity and pressure fields in the gap when the inertial terms in the conservation equation can be neglected.

   (d) Calculate the forces on the cylinder along the center line of the channel and perpendicular to it.

9. A two dimensional flow consists of a surface of length L moving at a small angle of inclination $\alpha \ll 1$ to a flat infinite surface as shown in figure 1. The length L is large compared to the distance between the two surfaces d_1 and d_2. The velocity of the surface is U in the horizontal direction. Assume that the flow is at *low Reynolds number*, so that inertial effects are negligible. For this system,



(a) Choose appropriate scalings for the lengths and velocities in the flow direction and cross stream directions. Scale the mass and momentum equations.

(b) Solve the equations, subject to the no slip condition at the two surfaces, to determine the velocity as a function of the pressure gradient.

(c) Determine the pressure gradient as a function of the total flux of fluid through the gap, using the condition that the flux at any horizontal position is a constant.

(d) How would you determine the value of the flux?

Figure 1