Chapter 11

Boundary layer theory

The simplest example of a boundary layer is the one formed at the surface of a flat plate in the limit of high Reynolds number. The configuration and co-ordinate system is shown in figure **??**. A flat plate of width L and infinitesimal thickness is placed along the $y-z$ plane from $x=0$ to $x=L$ in a fluid stream which has a uniform velocity U in the x direction far upstream of the plate. The mass and momentum conservation equations for the fluid at steady state are,

$$\nabla.\mathbf{u} = 0 \tag{11.1}$$

$$\rho\mathbf{u}.\nabla\mathbf{u} = -\nabla p + \mu\nabla^2\mathbf{u} \tag{11.2}$$

The boundary conditions for the fluid velocity field are the no-slip condition at the surface of the flat plate,

$$u_x = 0 \text{ at } y = 0 \text{and } x > 0 \tag{11.3}$$

$$u_y = 0 \text{ at } y = 0 \text{and } x > 0 \tag{11.4}$$

and the free-stream condition in the limit of large y,

$$u_x = U \text{ as } y \to \infty \tag{11.5}$$

There is an additional condition that the velocity is equal to the free-stream velocity before the fluid encounters the plate, for $x \leq 0$. Since convective transport is fast compared to diffusive transport ahead, the presence of the flat plate will not affect the fluid velocity upstream of the leading edge, and we require that

$$u_x = U \text{ for } x \leq 0 \tag{11.6}$$

As we shall see later, though the point $x = y = 0$ is a mathematical singularity, this does not complicate the solution procedure for the velocity profile.

The naive method for scaling the mass and momentum equations is to use scaled co-ordinates $x^* = (x/L)$, $y^* = (y/L)$ and $\mathbf{u}^* = (\mathbf{u}/U)$, since L and U are the only length and velocity scales in the problem. The appropriate scaled pressure in the high Reynolds number limit is $p^* = (p/\rho U^2)$. Expressed in terms of the scaled velocity and pressure, the mass and momentum conservation equations are,

$$\nabla^*.\mathbf{u}^* = 0 \tag{11.7}$$

$$\mathbf{u}^*.\nabla^*\mathbf{u}^* = -\nabla p^* + \mathrm{Re}^{-1}\nabla^2\mathbf{u}^* \tag{11.8}$$

In the limit of high Reynolds number, the viscous term (last term on the right side of equation 11.8) is expected to be small compared to the inertial terms, and the momentum conservation equation reduces to that for the potential flow past a flat surface. However, as we have seen in the previous chapter, the potential flow equations can satisfy only the normal velocity condition at the surface of the flat plate, and cannot satisfy the condition on the tangential velocity u_x in equation 11.3. This is a familiar problem we have faced in flows where convection is large compared to diffusion, and is caused by the fact that when we neglect the diffusive effects, the conservation equation is converted from a second order to a first order differential equation. Consequently, it is not possible to satisfy all the boundary conditions which were specified for the original second order differential equation.

The mass and momentum equations for the two-dimensional flow are,

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{11.9}$$

$$\rho\left(u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right) \tag{11.10}$$

$$\rho\left(u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right) \tag{11.11}$$

We now use the familiar strategy in boundary layer theory, which is to scale the cross-stream distance by a much smaller length scale, and adjust that length scale in order to achieve a balance between convection and diffusion. The dimensionless x co-ordinate is defined as $x^* = (x/L)$, while the dimensionless y co-ordinate is defined as $y^* = (y/l)$, where the length l is determined by a balance between convection and diffusion. In momentum boundary layers, it is also necessary to scale the velocity components and the pressure. In the streamwise direction, the natural scale for the velocity is the free-stream velocity U_∞, and so we define a scaled velocity in the x direction as $u_x^* = (u_x/U_\infty)$. The scaled velocity in the y direction is determined from the mass conservation condition, When the above equation is expressed in terms of the scaled variables $x^* = (x/L)$, $y^* = (y/l)$ and $u_x^* = (u_x/U_\infty)$, and multiplied throughout by (L/U_∞), we obtain,

$$\frac{\partial u_x^*}{\partial x^*} + \frac{L}{lU_\infty}\frac{\partial u_y}{\partial y^*} = 0 \tag{11.12}$$

The above equation 11.12 indicates that the appropriate scaled velocity in the y direction is $u_y^* = (u_y/(U_\infty l/L))$. Note that the magnitude of the velocity u_y in the cross-stream y direction, $(U_\infty l/L)$, is small compared to that in the stream-wise direction. This is a feature common to all boundary layers in incompressible flows.

Next, we turn to the x momentum equation, 11.10. When this is expressed in terms of the scaled spatial and velocity coordinates, and divided throughout by $(\rho U_\infty^2/L)$, we obtain,

$$u_x^*\frac{\partial u_x^*}{\partial x^*} + u_y^*\frac{\partial u_y^*}{\partial y^*} = -\frac{1}{\rho U_\infty^2}\frac{\partial p}{\partial x^*} + \frac{\mu}{\rho U_\infty L}\left(\frac{L^2}{l^2}\right)\left(\frac{\partial^2 u_x^*}{\partial y^{*2}} + \frac{l^2}{L^2}\frac{\partial^2 u_x^*}{\partial x^{*2}}\right) \tag{11.13}$$

The above equation indicates that it is appropriate to define the scaled pressure as $p^* = (p/\rho U_\infty^2)$. Also note that the factor $(\mu/\rho U_\infty L)$ on the right side of equation 11.13 is the inverse of the Reynolds number based on the free stream velocity and the length of the plate. In the right side of equation 11.13, we can also neglect the streamwise gradient $(\partial^2 u_x^*/\partial x_*^2)$, since this is multiplied by the factor $(l/L)^2$, which is small in the limit $(l/L) \ll 1$. With these simplifications, equation 11.13 reduces to,

$$u_x^*\frac{\partial u_x^*}{\partial x^*} + u_y^*\frac{\partial u_y^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \mathrm{Re}^{-1}\left(\frac{L^2}{l^2}\right)\frac{\partial^2 u_x^*}{\partial y^{*2}} \tag{11.14}$$

From the above equation, it is clear that a balance is achieved between convection and diffuion only for $(l/L) \sim \mathrm{Re}^{-1/2}$ in the limit of high Reynolds number. This indicates that the boundary layer thickness is $\mathrm{Re}^{-1/2}$ smaller than the length of the plate. Without loss of generality, we substitute

$l = \mathrm{Re}^{-1/2} L$ in equation 11.14, to get the scaled momentum equation in the streamwise direction,

$$u_x^* \frac{\partial u_x^*}{\partial x^*} + u_y^* \frac{\partial u_y^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u_x^*}{\partial y^{*2}} \tag{11.15}$$

Next, we analyse the momentum equation in the cross-stream direction, 11.11. This equation is expressed in terms of the scaled spatial co-ordinates, velocities and pressure, to obtain,

$$\frac{\rho U_\infty^2 l}{L^2} \left(u_x^* \frac{u_y^*}{x^*} + u_y^* \frac{\partial u_y^*}{\partial y^*} \right) = -\frac{\rho U_\infty^2}{l} \frac{\partial p_*}{\partial y^*} + \frac{\mu U_\infty}{lL} \left(\frac{\partial^2 u_y^*}{\partial y^{*2}} + \left(\frac{l}{L} \right)^2 \frac{\partial^2 u_y^*}{\partial x^{*2}} \right) \tag{11.16}$$

By examining all terms in the above equation, it is easy to see that the largest terms is the pressure gradient in the cross-stream direction. We divide throughout by the pre-factor of this term, and substitute $(l/L) = \mathrm{Re}^{-1/2}$, to obtain,

$$\mathrm{Re}^{-1} \left(u_x^* \frac{u_y^*}{x^*} + u_y^* \frac{\partial u_y^*}{\partial y^*} \right) = -\frac{\partial p_*}{\partial y^*} + \mathrm{Re}^{-1} \left(\frac{\partial^2 u_y^*}{\partial y^{*2}} + \mathrm{Re}^{-1} \frac{\partial^2 u_y^*}{\partial x^{*2}} \right) \tag{11.17}$$

In the limit $\mathrm{Re} \gg 1$, the above momentum conservation equation reduces to,

$$\frac{\partial p^*}{\partial y^*} = 0 \tag{11.18}$$

Thus, the pressure gradient in the cross-stream direction is zero in the leading approximation, and the pressure at any streamwise location in the boundary layer is the same as that in the free-stream at that same stream-wise location. This is a salient feature of the flow in a boundary layers. Thus, the above scaling analysis has provided us with the simplified 'boundary layer equations' 11.12, 11.15 and 11.18, in which we neglect all terms that are $o(1)$ in an expansion in the parameter $\mathrm{Re}^{-1/2}$. Expressed in dimensional form, these mass conservation equation is 11.9, while the approximate momentum conservation equations are,

$$\rho \left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u_x}{\partial y^2} \tag{11.19}$$

$$\frac{\partial p}{\partial y} = 0 \tag{11.20}$$

From equation 11.20, the pressure is only a function from the leading edge of the plate x, and not a function of the cross-stream distance y. Therefore, the pressure at a displacement x from the leading edge is independent of the normal distance from the plate y. However, in the limit $y \rightarrow \infty$, we know that the free stream velocity U_∞ is a constant, and the pressure is a constant independent of x. This implies that the pressure is also independent of x as well, and the term $(\partial p/\partial x)$ in equation 11.19 is equal to zero. With this, equation 11.19 simplifies to,

$$\rho \left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} \right) = \mu \frac{\partial^2 u_x}{\partial y^2} \tag{11.21}$$

This has to be solved, together with the mass conservation condition equation 11.9, to obtain the velocity profile.

We look for a similarity solution for the above equation, under the assumption that the stream function at a location (x, y) depends only on the distance x from the leading edge of the plate and the cross-stream distance y, and not on the total length of the plate. The justification for that momentum is being convected downstream by the flow, and so conditions at a trailing edge of the plate at $x = L$ should not affect the velocity profile upstream of this location. While scaling the spatial coordinates and velocities in equations 11.12, 11.15 and 11.18, we had used the dimensionless y coordinate

$$y^* = \frac{y}{Re^{-1/2}L} = \frac{y}{(\nu L/U_\infty)^{1/2}} \tag{11.22}$$

Since we have made the assumption that the only length scale in the problem is the distance from the leading edge x, it is appropriate to define the similarity varible using x instead of L in equation 11.22,

$$\eta = \frac{y}{(\nu x/U_\infty)^{1/2}} \tag{11.23}$$

This scaling implies that the thickness of the boundary layer at a distance x from the upstream edge of the plate is proportional to $(\nu x/U_\infty) = x\text{Re}_x^{-1/2}$, where $\text{Re}_x = (U_\infty x/\nu)$ is the Reynolds number on the distance from the upstream edge. It is appropriate to scale the velocity in the x direction by the free stream velocity U_∞, while the scaling for the velocity in the y direction is obtained by replacing L by x in equation 11.12,

$$u_x^* = \frac{u_x}{U_\infty} \tag{11.24}$$

$$u_y^* = \frac{u_y}{(\nu U_\infty/x)^{1/2}} \tag{11.25}$$

where u_x^* and u_y^* are only functions of the similarity variable η. It is convenient to express the velocity components in terms of the stream function $\psi(x, y)$ for an incompressible flow, since the mass conservation condition is identically satisfied when the velocity is expressed in terms of the stream function. The components of the velocity are related to the stream function by,

$$\begin{aligned} u_x^* &= \frac{1}{U_\infty}\frac{\partial \psi}{\partial y} \\ &= \frac{1}{(\nu x U_\infty)^{1/2}}\frac{\partial \psi}{\partial \eta} \end{aligned} \tag{11.26}$$

The above equation indicates that it is appropriate to define the scaled stream function ψ^*, often expressed in literature as $f(\eta)$,

$$\psi^*(\eta) = f(\eta) = \frac{\psi}{(\nu x U_\infty)^{1/2}} \tag{11.27}$$

where $f(\eta)$ is a dimensionless function of the similarity variable η. The streamwise velocity can then be expressed in terms of the scaled stream function as,

$$\begin{aligned} u_x &= \frac{\partial \psi}{\partial y} \\ &= U_\infty \frac{df}{d\eta} \end{aligned} \tag{11.28}$$

The cross-stream velocity is given by,

$$\begin{aligned} u_y &= -\frac{\partial \psi}{\partial x} \\ &= \frac{1}{2}\left(\frac{\nu U_\infty}{x}\right)^{1/2}\left(\eta\frac{df}{d\eta} - f\right) \end{aligned} \tag{11.29}$$

Equation 11.19 also contains derivatives of the streamwise velocity, which can be expressed in terms of the similarity variable η as,

$$\frac{\partial u_x}{\partial x} = -\frac{U_\infty \eta}{2x}\frac{d^2 f}{d\eta^2} \tag{11.30}$$

$$\frac{\partial u_x}{\partial y} = \frac{U_\infty}{(\nu x/U_\infty)^{1/2}} \frac{d^2 f}{d\eta^2} \quad (11.31)$$

$$\frac{\partial^2 u_x}{\partial y^2} = \frac{U_\infty^2}{\nu x} \frac{d^3 f}{d\eta^3} \quad (11.32)$$

Equations 11.29 to 11.32 are inserted into the equation 11.19 to obtain, after some simplification,

$$\frac{d^3 f}{d\eta^3} + \frac{1}{2} f \frac{d^2 f}{d\eta^2} = 0 \quad (11.33)$$

This is the 'Blasius boundary layer' equation for the stream function for the flow past a flat plate. This equation has to be solved, subject to the appropriate boundary conditions, which are as follows. At the surface of the plate, the no-slip condition requires that the velocity components u_x and u_y are zero. Since u_x is given by equation 11.28, the condition $u_x = 0$ at $y = 0$ reduces to,

$$\frac{df}{d\eta} = 0 \text{ at } y = 0 (\eta = 0) \quad (11.34)$$

Using equation 11.29 for the cross-stream velocity u_y, along with condition 11.34 at the surface, the condition $u_y = 0$ at $y = 0$ reduces to,

$$f = 0 \text{ at } y = 0 (\eta = 0) \quad (11.35)$$

Finally, we require that the velocity u_x is equal to the free stream velocity U_∞ in the limit $y \to \infty$. Using equation 11.29 for u_x, we obtain,

$$\frac{df}{d\eta} = 1 \text{ for } y \to \infty \quad \to \eta \to \infty \quad (11.36)$$

Further, we also require that the fluid velocity is equal to the free stream velocity U_∞ at the upstream edge of the plate $x = 0$ at any finite y. When expressed in terms of the similarity variable η, this boundary condition is identical to equation 11.34 for $y \to \infty$.

Equation 11.33, which is a third order equation, can be solved subject to three boundary conditions 11.34, 11.35 and 11.36 to obtain the variation of $f(\eta)$ with η. This equation cannot be solved analytically, but can be solved quite easily using numerical techniques. Figure **??** shows the solution for $(u_x/U_\infty) = (df/d\eta)$ as a function of η.

The shear stress at the surface is given by,

$$\begin{aligned}
\tau_{xy} &= \mu \left.\frac{\partial u_x}{\partial y}\right|_{y=0} \\
&= \frac{\mu}{(\nu x/U_\infty)^{1/2}} \left.\frac{du_x}{d\eta}\right|_{\eta=0} \\
&= \rho \left(\frac{\nu U_\infty^3}{x}\right)^{1/2} \left.\frac{d^2 f}{d\eta^2}\right|_{\eta=0} \qquad (11.37)
\end{aligned}$$

Therefore, the shear stress at the surface decreases as $x^{-1/2}$ as the downsteam distance x increases. The value of $(d^2 f/d\eta^2) = 0.664$ at $\eta = 0$ is obtained from the numerical solution of equation 11.33 for $f(\eta)$, and therefore the shear stress at the surface of the plate is,

$$\tau_{xy} = 0.332\rho \left(\frac{\nu U_\infty^3}{x}\right)^{1/2} \qquad (11.38)$$

The total force exerted on the plate, per unit length in the direction perpendicular to the flow, is determined by integrating equation the shear stress,

$$\begin{aligned}
F_x &= \int_0^L dx\, \tau_{xy} \\
&= 0.664\rho(\nu U_\infty^3 L)^{1/2} \qquad (11.39)
\end{aligned}$$

The ‘drag coefficient’ is defined as,

$$\begin{aligned}
C_D &= \frac{F_x}{(\rho U_\infty^2/2)L} \\
&= 1.328\mathrm{Re}_L^{-1/2} \qquad (11.40)
\end{aligned}$$

where Re_L is the Reynolds number based on the length of the plate and the free stream velocity U_∞.

11.1 Stagnation point flow

The stagnation point flow is encountered at the upstream stagnation point in the flow past bluff bodies such as solid spheres, where the fluid velocity is perpendicular to the surface of the object. Over length scales small compared to the radius of curvature of the object, the flow can be approximated by a fluid stream incident on a flat plate as shown in figure **??**. In the potential flow approximation, the stream function and the velocity components, in the co-ordinate system shown in figure **??**, are

$$\psi = \dot{\gamma} x y \tag{11.41}$$

$$u_x = \dot{\gamma} x \tag{11.42}$$

$$u_y = -\dot{\gamma} y \tag{11.43}$$

where $\dot{\gamma}$ is the strain rate $(\partial u_x/\partial y) = -(\partial u_y/\partial x)$, which has units of inverse time. The pressure in the potential flow is,

$$\begin{aligned} p &= -\frac{\rho(u_x^2 + u_y^2)}{2} \\ &= -\frac{\rho\dot{\gamma}^2(x^2+y^2)}{2} \end{aligned} \tag{11.44}$$

The boundary conditions are the no-slip condition at the surface of the plate,

$$u_x = \frac{\partial \psi}{\partial y} = 0 \text{ at } y = 0 \tag{11.45}$$

$$u_y = -\frac{\partial \psi}{\partial x} = 0 \text{ at } y = 0 \tag{11.46}$$

while the condition that the velocity is equal to the potential flow solution at a large distance from the plate can be written as,

$$\psi \to \dot{\gamma} x y \text{ for } y \to \infty \tag{11.47}$$

The potential flow solution for the velocity 11.42 does not satisfy the boundary condition at the surface of the plate, and so it is necessary to postulate a boundary layer very close to the surface where viscous effects become important. In this boundary layer, it is necessary to solve the boundary layer equations 11.9, 11.19 and 11.20, subject to the appropriate boundary conditions, to obtain the velocity profile.

The thickness of the boundary layer can be inferred as follows. In the flow past a flat plate, the boundary layer thickness at a distance x from the upstream edge was found to be proportional to $(\nu x/U_\infty)^{1/2}$. In the stagnation point flow, the velocity U_∞ in the potential flow outside the boundary layer is not a constant, but is equal to $\dot{\gamma}x$. Therefore, we would expect the boundary layer thickness to be proportional to $(\nu/\dot{\gamma})^{1/2}$. The boundary layer approximation will be valid only when this thickness is small compared to the distance x from the plane of symmetry of the flow, $(\nu/\dot{\gamma})^{1/2} \ll x$, or for $(\dot{\gamma}x^2/\nu) \gg 1$. The dimensionless parameter $(\dot{\gamma}x^2/\nu)$ is the Reynolds number based on the strain rate and the distance x from the plane of symmetry of the flow, and the boundary layer approximation is valid when this Reynolds number is large.

We define the similarity variable $\eta = y/(\nu/\dot{\gamma})^{1/2}$, since the boundary layer thickness is $(\nu/\dot{\gamma})^{1/2}$. If we express the outer potential flow solution in terms of the similarity variable, we obtain,

$$\psi = \nu^{1/2}\dot{\gamma}^{1/2}x\eta \tag{11.48}$$

Therefore, within the boundary layer, it is appropriate to assume a stream function of the form,

$$\psi = \nu^{1/2}\dot{\gamma}^{1/2}xf(\eta) \tag{11.49}$$

where $f(\eta)$ is a dimensionless function of the similarity variable η. The pressure gradient in the potential flow in the x direction is given by,

$$\frac{\partial p}{\partial x} = -\rho\dot{\gamma}x \tag{11.50}$$

Since the pressure gradient $(\partial p/\partial y)$ in the boundary layer is zero, the pressure gradient $(\partial p/\partial x)$ in the boundary layer is also given by equation 11.50.

Equation 11.19 for the x momentum has to be solved, using the form 11.49 for the stream function, in order to determine the similarity solution $f(\eta)$. The velocity and velocity gradients in equation 11.19, expressed in terms of $f(\eta)$, are,

$$\begin{aligned} u_x &= \frac{\partial\psi}{\partial y} = \dot{\gamma}x\frac{df}{d\eta} && (11.51) \\ u_y &= -\frac{\partial\psi}{\partial x} = -\dot{\gamma}^{1/2}\nu^{1/2}f \\ \frac{\partial u_x}{\partial x} &= \dot{\gamma}\frac{df}{d\eta} && (11.52) \\ \frac{\partial u_x}{\partial y} &= \frac{\dot{\gamma}^{3/2}x}{\nu^{1/2}}\frac{d^2f}{d\eta^2} && (11.53) \\ \frac{\partial^2 u_x}{\partial y^2} &= \frac{\dot{\gamma}^2x}{\nu}\frac{d^3f}{d\eta^3} && (11.54) \end{aligned}$$

The above velocity and velocity gradients are inserted into the x momentum balance equation 11.19, and divided throughout by $(\dot{\gamma}x)$, to obtain,

$$\frac{d^3f}{d\eta^3} - \left(\frac{df}{d\eta}\right)^2 + f\frac{d^2f}{d\eta^2} + 1 = 0 \tag{11.55}$$

The boundary conditions for $f(\eta)$ can be obtained from the boundary conditions on the stream function 11.45 and 11.46 at the surface of the plate $y = 0$ $(\eta = 0)$, and the condition 11.47 in the limit $y \to \infty$ $(\eta \to \infty)$.

$$f(\eta) = 0 \text{ at } \eta = 0 \tag{11.56}$$

$$\frac{df}{d\eta} = 0 \text{ at } \eta = 0 \tag{11.57}$$

$$f(\eta) \to \eta \text{ for } \eta \to \infty \tag{11.58}$$

The shear stress at the surface is given by,

$$\tau_{xy} = \mu \left.\frac{\partial u_x}{\partial y}\right|_{y=0} \tag{11.59}$$

$$= \rho(\nu^{1/2}\dot{\gamma}^{3/2}x^{1/2}) \left.\frac{d^2f}{d\eta^2}\right|_{\eta=0} \tag{11.60}$$

Therefore, the shear stress at the surface increases as $x^{1/2}$ as the downsteam distance x increases.

(The value of $(d^2f/d\eta^2) = 0.664$ at $\eta = 0$ is obtained from the numerical solution of equation **??** for $f(\eta)$, and therefore the shear stress at the surface of the plate is,

$$\tau_{xy} = 0.332\rho\left(\frac{\nu U_\infty^3}{x}\right)^{1/2} \tag{11.61}$$

The total force exerted on the plate, per unit length in the direction perpendicular to the flow, is determined by integrating equation the shear stress,

$$F_x = \int_0^L dx\, \tau_{xy}$$

$$= 0.664\rho(\nu U_\infty^3 L)^{1/2} \tag{11.62}$$

The 'drag coefficient' is defined as,

$$C_D = \frac{F_x}{(\rho U_\infty^2/2)L}$$

$$= 1.328 Re_L^{-1/2} \tag{11.63}$$

where Re_L is the Reynolds number based on the length of the plate and the free stream velocity U_∞.)

11.2 Falkner-Skan solutions

In the previous two sections, we obtained boundary layer solutions for two specific potential flow velocity profiles in the flow past a flat plate. This raises the question of whether it is possible to obtain boundary layer solutions for all flows past solid surfaces in the high Reynolds number limit. It turns out that it is possible to obtain boundary layer solutions only for a specific class of flows past flat solid surfaces, and this class of solutions is called the 'Falkner-Skan' boundary layer solutions.

The configuration consists of a two-dimensional flow past a solid object, and we assume that the potential flow velocity profile is knows. Near the surface, the no-slip boundary conditions cannot be satisfied by the potential flow solution, and so it is necessary to postulate the presence of a 'boundary layer' near the surface whose thickness is small compared to the characteristic length of the object. A local co-ordinate system (x, y) is used, where x is along the surface and y is perpendicular to the surface. The normal velocity at the surface of the object is zero in the potential flow approximation, though the tangential velocity is, in general, non-zero. We assume that the tangential velocity profile is given by $U(x)$ in the boundary layer, which corresponds to the limit $y \to 0$ for the potential flow solution.

The pressure gradient in the x direction in equation 11.19 can be determined in terms of the potential flow velocity $U(x)$ as follows. Equation 11.20 indicates that the pressure gradient in the cross-stream direction is zero, so that the pressure is independent of y. Therefore, the pressure at any location is a function of x alone, and $p(x, y) = p(x, y \to \infty)$. However, the limit $y \to \infty$ corresponds to the free stream, where the flow is not affected by the presence of the surface at $y = 0$. The free-stream velocity profile satisfies the potential flow equations, and the pressure p_∞ is related to the free stream velocity by the Bernoulli equation,

$$p_\infty + \frac{\rho U^2}{2} = \text{Constant} \tag{11.64}$$

Since the pressure depends only on x and not on y, the pressure at any location $p(x,y) = p_\infty(x)$, and the pressure gradient in the x direction is,

$$\frac{\partial p}{\partial x} = \frac{\partial p_\infty}{\partial x} = -\rho U \frac{dU}{dx} \tag{11.65}$$

When this is inserted into the momentum conservation equation 11.19, we obtain,

$$\rho \left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} \right) = \rho U \frac{\partial U}{\partial x} + \mu \frac{\partial^2 u_x}{\partial y^2} \tag{11.66}$$

The boundary conditions at the surface of are given by 11.3 and 11.4,

$$u_x = 0 \text{ at } y = 0 \tag{11.67}$$

$$u_y = 0 \text{ at } y = 0 \tag{11.68}$$

while the boundary condition in the limit $y \to \infty$ is given by equation 11.5.

$$u_x = U(x) \text{ for } y \to \infty \tag{11.69}$$

The fluid velocity and pressure fields are governed by equations 11.9, 11.20 and 11.66. We assume that the solution can be expressed in terms of a similarity variable $\eta = (y/\delta(x))$, where the boundary layer thickness $\delta(x)$ is a function of x. Note that $\delta(x) \ll L$ in the high Reynolds number limit, where L is the characteristic length scale of the flow. The form for the velocity u_x is chosen to be,

$$u_x = U(x) \frac{df}{d\eta} \tag{11.70}$$

Note that this is of the same form as that for the flow past a flat plate in equation 11.27, and for the stagnation point flow in equation 11.49 where $U(x) = \dot{\gamma} x$. The stream function can be obtained by integrating equation 11.70 with respect to y,

$$\begin{aligned} \psi &= \int_0^y dy\, u_x \\ &= \delta(x) U(x) \int_0^\eta d\eta' \, \frac{df}{d\eta} \\ &= \delta(x) U(x) f(\eta) \end{aligned} \tag{11.71}$$

The velocity u_y in the y direction is,

$$\begin{aligned} u_y &= -\frac{\partial \psi}{\partial x} \\ &= -\frac{d(\delta(x)U(x))}{dx} f(\eta) + \frac{U(x) y}{\delta(x)^2} \frac{d\delta(x)}{dx} \frac{df}{d\eta} \end{aligned} \tag{11.72}$$

The velocity gradients in the x momentum equation can now be expressed in terms of $f(\eta)$,

$$\frac{\partial u_x}{\partial x} = \frac{dU(x)}{dx}\frac{df}{d\eta} - \frac{U(x)y}{\delta(x)^2}\frac{d\delta(x)}{dx}\frac{d^2 f}{d\eta^2} \quad (11.73)$$

$$\frac{\partial u_x}{\partial y} = \frac{U(x)}{\delta(x)}\frac{d^2 f}{d\eta^2} \quad (11.74)$$

$$\frac{\partial^2 u_x}{\partial^2 y} = \frac{U(x)}{\delta(x)^2}\frac{d^3 f}{d\eta^3} \quad (11.75)$$

These are inserted into the x momentum equation 11.66, and the result is divided throughout by $(\nu U(x)/\delta(x)^2)$, to obtain,

$$\frac{d^3 f}{d\eta^3} - \frac{\delta^2}{\nu}\frac{dU(x)}{dx}\left(\frac{df}{d\eta}\right)^2 + \frac{U(x)\delta(x)}{\nu}\eta\frac{d\delta(x)}{dx}\frac{df}{d\eta}\frac{d^2 f}{d\eta^2}$$
$$+\frac{\delta^2}{\nu}\frac{df}{d\eta}\left(\frac{d(\delta(x)U(x))}{dx}f(\eta) + \frac{U(x)y}{\delta(x)^2}\frac{d\delta(x)}{dx}\frac{df}{d\eta}\right) = 0 \quad (11.76)$$

Problems:

1. Consider the uniform flow of a fluid past a flat plate of infinite extent in the $x_1 - x_3$ plane, with the edge of the plate at $x_1 = 0$. The Reynolds number based on the fluid velocity and the length of the plate is large. At a large distance from the plate, the fluid has a uniform velocity U_1 in the x_1 direction and U_3 in the x_3 direction. All velocities are independent of the x_3 direction, and the velocities U_1 and U_3 are uniform and independent of position in the outer flow,

   (a) Write down the equations of motion in the x_1, x_2 and x_3 directions. If the thickness of the boundary layer δ is small compared to the length of the plate L, find the leading order terms in the conservation equations.

   (b) What is the similarity form of the equation for the momentum equations for the velocity u_1? Make use of the similarity forms for boundary layer flows derived in class. **Do not try to solve the equation.**

   (c) Find the solution for the velocity u_3 in terms of the solution for u_1.

(d) If the velocity U_1 in the outer flow is a constant, but the velocity $U_3 = U_{30} + U_{31} x_1$, what is the form of the equation for the velocity u_3? Is a similarity solution possible?

2. Consider the Blasius boundary layer flow of a fluid past a flat plate of length L in the high Reynolds number limit, where x and y are the streamwise and cross-stream directions respectively. The velocity profile is given by the Blasius boundary layer profile. The temperature of the fluid at a large distance from the plate is T_0. For $x < 0$, the temperature of the plate is T_0, while for $x > 0$, the plate is at a higher temperature $T_1 > T_0$. The equation for the temperature field in the fluid is:
$$\frac{\partial T}{\partial t} + \mathbf{u}.\nabla T = \alpha \nabla^2 T$$
where $\mathbf{u}$ is the fluid velocity. The above equation gives a balance between the convective and diffusive transport of heat in the fluid.

   (a) Scale the coordinates in the temperature equation, and determine the equivalent Peclet number for this problem.

   (b) Assume that the temperature field can be expressed in terms of the similarity variable using a suitable form, and derive a similarity equation for the temperature field, which contains the Prandtl number, the ratio of the Peclet and Reynolds numbers.

   (c) If this number is large, one would expect the conduction to be confined to a thin layer near the surface of the plate. Scale the coordinates in the heat balance equation in this case, and determine the boundary layer thickness as a function of the Prandtl number.

   (d) What is the scaling of the flux as a function of the Prandtl and Reynolds numbers?