Chapter 1

Potential flow:

1.1 General formulation

Inviscid and irrotational flows in the limit of high Reynolds number are referred to as 'potential' or 'ideal' flows. The term 'inviscid' refers to flows where viscous forces are small compared to inertial forces, so that the fluid viscosity can be neglected in comparison to fluid inertia. 'Potential' or 'ideal' flows are a class of inviscid flows in which the vorticity ω, which is the curl of the velocity vector, is zero, i. e.

$$\omega_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j} = 0 \tag{1.1}$$

Since the curl of the velocity is zero, the velocity can be expressed as the gradient of a potential ϕ,

$$u_i = \frac{\partial \phi}{\partial x_i} \tag{1.2}$$

and hence the name 'potential flow'.

Using equation 1.2 for the velocity field, the Navier-Stokes mass and momentum equations can be written in terms of the velocity potential. The mass conservation equation, expressed in terms of the velocity potential, is,

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial^2 \phi}{\partial x_i^2} = 0 \tag{1.3}$$

Thus, the mass conservation equation simply states that the velocity potential satisfies the Laplace equation, $\nabla^2 \phi = 0$. Therefore, for potential flow we can use all the techiques developed for solving the Laplace equation earlier. The momentum conservation equation an inviscid flow is,

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{f_i}{\rho} \tag{1.4}$$

where f_i is the body force per unit volume acting on the fluid. The second term on the right side of the above equation can be simplified for an irrotational flow in which the vorticity is zero. Consider the cross product of the velocity and vorticity, $\mathbf{u} \times \omega = \mathbf{u} \times \nabla \times \mathbf{u}$, which can be written in indicial notation as,

$$\begin{aligned} \epsilon_{ijk} u_j \epsilon_{klm} \frac{\partial u_m}{\partial x_l} &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) u_j \frac{\partial u_m}{\partial x_l} \\ &= u_j \frac{\partial u_i}{\partial x_j} - u_i \frac{\partial u_i}{\partial x_j} \\ &= u_j \frac{\partial u_i}{\partial x_j} - \frac{1}{2} \frac{\partial (u_i^2)}{\partial x_j} \end{aligned} \tag{1.5}$$

For an irrotational flow, the left side of the above equation is zero since the vorticity $\nabla \times \mathbf{u}$ is zero. Therefore,

$$u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{2} \frac{\partial (u_i^2)}{\partial x_j} \tag{1.6}$$

Using the above substitution for the second term on the left side of equation 1.4, and also expressing the first term on the left in terms of the velocity potential using equation 1.2, we obtain,

$$\frac{\partial}{\partial t} \frac{\partial \phi}{\partial x_i} + \frac{1}{2} \frac{\partial (u_i^2)}{\partial x_i} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} - \frac{f_i}{\rho} = 0 \tag{1.7}$$

If the body force f_i is conservative, it can be expressed as the gradient of a potential V as follows,

$$f_i = -\frac{\partial V}{\partial x_i} \tag{1.8}$$

When this is inserted into equation 1.7, and simplified, we obtain,

$$\frac{\partial}{\partial x_i} \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} u_i^2 + \frac{p}{\rho} + \frac{V}{\rho} \right) = 0 \tag{1.9}$$

Since all components of the gradient of the term in brackets is zero in the above equation, the term in brackets has to be equal to a constant,

$$\left(\frac{\partial \phi}{\partial t} + \frac{1}{2} u_i^2 + \frac{p}{\rho} + \frac{V}{\rho} \right) = \frac{p_0}{\rho} \tag{1.10}$$

where the constant p_0 is the reference pressure at a location where the velocity and potential V are zero. In a gravitational field, the potential is equal to $\rho g z$, where z is the height above the zero-potential reference. Equation 1.10 can then be written as,

$$\left(\frac{\partial \phi}{\partial t} + \frac{1}{2} u_i^2 + \frac{p}{\rho} + g z \right) = \frac{p_0}{\rho} \tag{1.11}$$

Equation 1.11 is referred to as the 'Bernoulli' equation for the potential flow.

An important point to note is that in potential flows, it is possible to satisfy only the normal velocity and stress conditions at a surface, and it is not possible to satisfy the tangential velocity and stress conditions. This is because we have neglected the viscous terms in the momentum conservation equation 1.4, which contain the second spatial derivatives of the velocity field. Consequently, we have reduced the equations from a second order to a first order equation in the spatial coordinates. And therefore, it is possible to satisfy only the normal velocity and stress boundary conditions at the surface. If we neglect viscous effects, the only contribution to the stress is the isotropic contribution due to the pressure,

$$\tau_{ij} = -p\delta_{ij} \tag{1.12}$$

It should be noted that the Laplace equation 1.3 for the velocity potential is a linear equation, and if normal velocity boundary conditions are prescribed at the bounding surface, the potential is a linear function of the velocity. If the velocity of all surfaces is changed by a constant factor, the potential and the velocity at all points in the fluid also change by the same factor. In addition, the solutions for the velocity potential are 'quasi-static' for imposed normal velocities of the bounding surfaces, because the potential at a given instant in time depend only on the velocity of the bounding surfaces at that instant. However, the Bernoulli equation 1.10 for the pressure is a non-linear equation, since it has a contribution proportional to the square of the velocity, and is also explicitly dependent on time, since it contains a term proportional to the time derivative of the potential. Therefore, the solution for the velocity potential under imposed normal stresses at the bounding surfaces is not linear or quasi-static.

There are some general results that can be derived for potential flows. The first is that the kinetic energy in a potential flow can be expressed as a function of the potential and normal velocity at the bounding surfaces. The kinetic energy of the fluid in a volume V with bounding surfaces S is defined as,

$$\begin{aligned}
\mathrm{KE} &= \frac{\rho}{2}\int_V dV u_j^2 \\
&= \frac{\rho}{2}\int_V dV \left(\frac{\partial\phi}{\partial x_j}\frac{\partial\phi}{\partial x_j}\right) \\
&= \frac{\rho}{2}\int_V dV \left(\frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_j}\right) - \phi\frac{\partial^2\phi}{\partial x_j^2}\right)
\end{aligned} \tag{1.13}$$

where integration by parts has been used in the final step. Since the potential satisfies the Laplace equation 1.3, the second term in the integrand on the right side of equation 1.13 is zero, and the expression for the kinetic energy reduces to,

$$\begin{aligned}
\mathrm{KE} &= \frac{\rho}{2}\int_V dV \frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_j}\right) \\
&= \frac{\rho}{2}\int_S \phi u_j n_j
\end{aligned} \tag{1.14}$$

where **n** is the outward unit normal at the surfaces bounding the fluid. Therefore, the kinetic energy of the fluid can be expressed only in terms of the potential and normal velocity at the bounding surfaces. It follows that if the normal velocity at all bounding surfaces is zero, then the kinetic energy is zero, which implies that the fluid velocity is zero throughout the domain.

It is possible to prove a 'uniqueness theorem' for potential flows, which states that the potential flow solution is unique if the normal velocity boundary conditions are specified on all boundaries. To prove this, we first assume that the potential flow solution is not unique, and that there are two potential flow solutions, $\mathbf{u}$ and $\mathbf{u}^*$, which satisfy the same normal velocity boundary conditions, $u_i n_i = u_i^* n_i$ on the bounding surfaces of the flow. Consider the integral over the fluid domain V,

$$I = \int_V dV (u_i^* - u_i)(u_i^* - u_i) \tag{1.15}$$

Since the integrand is always positive, the integral also has to be positive. For a potential flow, this integral can be rewritten by expressing the velocity in terms of the velocity potential, and using integration by parts as follows.

$$\begin{aligned} I &= \int_V dV (u_i^* - u_i) \frac{\partial(\phi^* - \phi)}{\partial x_i} \\ &= \int_V dV \frac{\partial}{\partial x_i}((u_i^* - u_i)(\phi^* - \phi) - \int_V dV (\phi^* - \phi) \frac{\partial(u_i^* - u_i)}{\partial x_i} \end{aligned} \tag{1.16}$$

The second integral on the right side of equation 1.16 is zero due to the incompressibility condition, while the first integral can be expressed as a surface integral using the divergence theorem,

$$\begin{aligned} I &= \int_S dS (\phi^* - \phi)(u_i^* - u_i) n_i \\ &= 0 \end{aligned} \tag{1.17}$$

Here, we have used the condition that the two velocity fields satisfy the same normal velocity conditions on the boundaries, so that $(u_i^* - u_i)n_i$ is zero on all boundaries.

We had earlier assumed that $\mathbf{u}^*$ and $\mathbf{u}$ are different velocity fields, in which case the integral I has to be positive. However, we have proved in equation 1.17 that I is zero if the normal velocity fields at all surfaces are equal. Therefore, this implies, that it is not possible to have two different potential flow solutions which satisfy the same normal velocity boundary conditions at all the bounding surfaces, and the solutions for the potential flow equations is unique if the normal velocity conditions are specified at all boundaries.

Another general result is the 'minimum energy theorem', which states that the kinetic energy of a potential flow is less than or equal to the kinetic energy of any other flow that satisfies the same normal velocity boundary conditions at the bounding surfaces. Let us consider two velocity profiles, u_i and u_i^*, both of which satisfy the same normal velocity boundary conditions at the bounding surfaces ($u_i n_i = u_i^* n_i$). The velocity field u_i is a solution of the potential flow equations, whereas the velocity field u_i^* is not necessarily a solution of the potential flow condition, though it satisfies the mass conservation condition $(\partial u_i^*/\partial x_i) = 0$. The difference in the kinetic energies of the two flows is,

$$\begin{aligned}
\text{KE}^* - \text{KE} &= \frac{\rho}{2}\int_V dV(u_i^{*2} - u_i^2) \\
&= \frac{\rho}{2}\int_V dV\left((u_i^* - u_i)^2 + 2u_i(u_i^* - u_i)\right) \\
&= \frac{\rho}{2}\int_V dV(u_i^* - u_i)^2 + \rho\int_V dV u_i(u_i^* - u_i) \\
&= \frac{\rho}{2}\int_V dV(u_i^* - u_i)^2 + \rho\int_V dV \frac{\partial\phi}{\partial x_i}(u_i^* - u_i) \\
&= \frac{\rho}{2}\int_V dV(u_i^* - u_i)^2 + \rho\int_V dV\left(\frac{\partial}{\partial x_i}(\phi(u_i^* - u_i))\right) - \phi\frac{\partial(u_i^* - u_i)}{\partial x_i}
\end{aligned} \tag{1.18}$$

where integration by parts has been used in the final step. The second term on the second integral on the right side of equation 1.16 is zero because the velocities u_i and u_i^* satisfy the incompressibility condition. Finally, the right side of equation 1.16 can be simplified using the divergence theorem,

$$\begin{aligned}
\text{KE}^* - \text{KE} &= \frac{\rho}{2}\int_V dV(u_i^* - u_i)^2 + \rho\int_V dV\left(\frac{\partial}{\partial x_i}(\phi(u_i^* - u_i))\right) \\
&= \frac{\rho}{2}\int_V dV(u_i^* - u_i)^2 + \rho\int_S dS n_i\phi(u_i^* - u_i))
\end{aligned} \tag{1.19}$$

Note the surface integral on the right side of equation 1.19 is zero, since it was assumed that the velocities u_i^* and u_i satisfy the same normal velocity boundary conditions at the surface. Therefore, the kinetic energy of the velocity field u_i^* that does not necessarily satisfy the potential flow equations is always equal to or greater than that of velocity field u_i which does satisfy the potential flow equations,

$$\begin{aligned}
\text{KE}^* - \text{KE} &= \frac{\rho}{2}\int_V dV(u_i^* - u_i)^2 \\
&\geq 0
\end{aligned} \tag{1.20}$$

1.2 Three-dimensional potential flows

1.2.1 Motion of a sphere in an infinite fluid

The simplest three-dimensional potential flow is the motion of a sphere in a fluid that it at rest at a large distance from the sphere. The configuration consists of a sphere of radius R moving with a velocity $\mathbf{U}$ in a fluid that is at rest at a large distance from the sphere, as shown in figure 1.1. The normal velocity

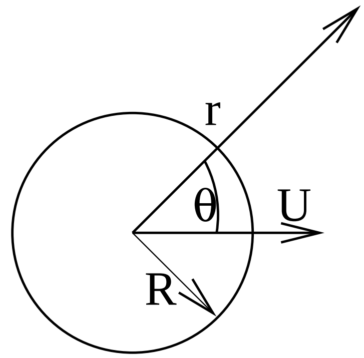

Figure 1.1: Sphere of radius R moving with a velocity U in a fluid which is at rest at large distance from the sphere.

boundary condition for the fluid velocity u_i at the surface of the sphere is,

$$u_i n_i = U_i n_i \tag{1.1}$$

The equation for the velocity potential, 1.3, has to be solved subject to the boundary conditions 1.1 in order to determine the velocity field. Since equation 1.3 is linear, the potential is a linear function of the velocity of the particle. In addition, the potential is a solution of the Laplace equation which decays to zero at a large distance from the sphere, so it is a linear combination of the spherical harmonic solutions. It is possible to construct only one scalar function which is linear in the velocity and in one of the spherical harmonics,

$$\phi = \frac{A U_j x_j}{r^3} \tag{1.2}$$

where A is a constant. The velocity field is then given by,

$$\begin{aligned} u_i &= \frac{\partial \phi}{\partial x_i} \\ &= A U_j \left(\frac{\delta_{ij}}{r^3} - \frac{3 x_i x_j}{r^5} \right) \end{aligned} \tag{1.3}$$

The constant A is determined from the normal velocity boundary condition 1.1. The normal fluid velocity at the surface is given by,

$$\begin{aligned} u_i n_i &= \frac{u_i x_i}{r} \\ &= -\frac{2 A U_j x_j}{r^4} \\ &= -\frac{2 A U_j n_j}{r^3} \end{aligned} \tag{1.4}$$

Equation the normal velocity 1.4 to $U_j n_j$ at the surface $r = R$, we obtain the solution $A = -(R^3/2)$. Therefore, the final solution for the velocity potential which satisfies the boundary conditions at the surface is,

$$\phi = -\frac{R^3 U_j x_j}{2r^3} \tag{1.5}$$

It is convenient to express the potential in a spherical co-ordinate system with co-ordinates (r, θ, ϕ), in which the azimuthal angle θ is measured from the direction of the velocity $\mathbf{U}$. In this co-ordinate system, the potential is independent of the meridional angle, and is given by,

$$\phi = -\frac{R^3 U \cos(\theta)}{2r^2} \tag{1.6}$$

The components of the velocity in the r and θ directions are,

$$\begin{aligned} u_r &= \frac{\partial \phi}{\partial r} \\ &= -\frac{R^3 U \cos(\theta)}{r^3} \end{aligned} \tag{1.7}$$

$$\begin{aligned} u_\theta &= \frac{1}{r}\frac{\partial \phi}{\partial \theta} \\ &= \frac{R^3 U \sin(\theta)}{2r^3} \end{aligned} \tag{1.8}$$

The total kinetic energy due to the fluid flow can be calculated using equation 1.14. Note that the unit normal $\mathbf{n}$ is the outward unit normal at the bounding surfaces of the fluid. The integral in equation 1.14 has no contribution due to the surface at a large distance from the sphere, since the velocity decreases to zero. At the surface of the fluid, the outward unit normal to the fluid is directed into the sphere, and therefore $n_i = -(x_i/r)$ at the surface of the sphere. In addition, the normal velocity of the fluid at the surface $u_j n_j$ is equal to the normal velcoity of the sphere $U_j n_j$, and therefore the kinetic energy is given by,

$$\begin{aligned} \mathrm{KE} &= -\frac{\rho}{2}\int_{S_{\text{sphere}}} dS \frac{\phi U_j x_j}{r} \\ &= \frac{\rho}{2}\int_{S_{\text{sphere}}} dS \frac{R^3 U_k x_k}{r^3}\frac{U_j x_j}{r} \\ &= \frac{\rho R^3 U_j U_k}{4}\int_{S_{\text{sphere}}} dS \frac{x_j x_k}{r^4} \end{aligned} \tag{1.9}$$

The surface integral in equation 1.9 can easily be evaluated,

$$\int_{S_{\text{sphere}}} dS \frac{x_j x_k}{r^4} = \frac{4\pi}{3} \tag{1.10}$$

Therefore the final expression for the kinetic energy due to the fluid motion is,

$$\mathrm{KE} = \left(\frac{2\pi R^3 \rho}{3}\right)\frac{U_j^2}{2} \tag{1.11}$$

If the kinetic energy of due to the moving fluid is expressed as $(M_a U_j^2/2)$, where M_a is the 'added mass' which has to be moved in addition to the mass of the moving fluid, then the added mass is given by

$$M_a = \frac{2\pi R^3 \rho}{3} \tag{1.12}$$

It can easily be seen that the added mass is equal to half the mass of fluid that is displaced by the sphere.

The pressure exerted on the fluid at the surface of the sphere can be evaluated,

$$p = p_0 - \rho\left(\frac{\partial \phi}{\partial t} + \frac{u_j^2}{2}\right) \tag{1.13}$$

It is necessary to exercise some care in calculating the time derivative of the potential in the above equation. Though the velocity of the sphere is a constant, the potential in a fixed reference frame is not a constant. This is because the radius vector from the center of the sphere to a fixed location in space changes as the sphere moves, and consequently the potential at this location also changes. If $\mathbf{x}_0$ is the location of the center of the particle, and $\mathbf{x}$ is the observation point, the equation for the potential 1.2 can be written as,

$$\phi = -\frac{R^3 U_j}{2}\frac{(x_j - x_{0j})}{|\mathbf{x} - \mathbf{x}_0|^3} \tag{1.14}$$

Since the sphere is moving with velocity $\mathbf{U}$, the rate of change of potential is,

$$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial U_j}\frac{dU_j}{dt} + \frac{\partial \phi}{\partial x_{0k}}\frac{dx_{0k}}{dt} \tag{1.15}$$

The derivative of the potential with respect to x_{0k} can be written as the negative of the derivative with respect to x_k in the above equation, because the potential is only a function of the distance $(\mathbf{x} - \mathbf{x}_0)$. Therefore, the change in $\mathbf{x} - \mathbf{x}_0$ due to a displacement of $\Delta\mathbf{x}$ in the position $\mathbf{x}_0$ is identical to that due to a displacement of $-\Delta\mathbf{x}$ in the position $\mathbf{x}$. Therefore, $(\partial\phi/\partial x_{0k}) = -(\partial\phi/\partial x_k)$, and equation 1.16 reduces to,

$$\begin{aligned}\frac{\partial \phi}{\partial t} &= \frac{\partial \phi}{\partial U_j}\frac{dU_j}{dt} - \frac{\partial \phi}{\partial x_k}U_k \\ &= \frac{\partial \phi}{\partial U_j}\frac{dU_j}{dt} - U_k u_k \end{aligned} \tag{1.16}$$

This is inserted into the equation for the pressure to obtain,

$$p = p_0 - \rho\left(\frac{u_j^2}{2} - u_j U_j \frac{\partial \phi}{\partial U_j}\frac{dU_j}{dt}\right) \tag{1.17}$$

In the above equation, we have returned to a co-ordinate system in which the origin is located at the center of the particle. The radial component of the fluid velocity, u_r, at the surface is zero, and the only contribution to the kinetic energy is due to the tangential velocity at the surface, u_θ (equation 1.8). Therefore, the pressure at the surface $r = R$ is given by,

$$\begin{aligned} p &= p_0 - \rho \left(\frac{u_\theta^2}{2} + U \sin(\theta) u_\theta - R \cos(\theta) \frac{dU}{dt} \right) \\ &= p_0 - \rho U^2 \left(\frac{5 \sin(\theta)^2}{16} - \frac{R}{U^2} \cos(\theta) \frac{dU}{dt} \right) \end{aligned} \tag{1.18}$$

In equation 1.19, we have substituted $U_\theta = -U \sin(\theta)$ and $u_\theta = (U \sin(\theta)/2)$ at $r = R$.

The force exerted on the sphere can be calculated by integrating the pressure over the surface of the sphere,

$$\begin{aligned} F_i &= \int_{S_{\text{sphere}}} dS \tau_{ij} n_j \\ &= \int_{S_{\text{sphere}}} dS p n_i \end{aligned} \tag{1.19}$$

By symmetry, the force perpendicular to the velocity is zero. The force along the direction of the velocity is,

$$F_z = \int_{S_{\text{sphere}}} dS p n_z \tag{1.20}$$

where F_z is the force in the direction of the velocity vector, and n_z is the component of the unit normal to the sphere in the direction of the velocity vector. For the steady flow of a particle with $(dU/dt) = 0$, it can easily be shown that the force F_i is identically zero due to symmetry considerations. The pressure p is a symmetric function of θ about the mid-plane cutting the sphere perpendicular to the direction of flow at $\theta = (\pi/2)$. However, the unit normal $n_z = \cos(\theta)$ is anti-symmetric about the plane $\theta = (\pi/2)$, and therefore the integral of the pressure and n_z over the surface of the sphere is identically zero. This result, that the steady motion of a sphere through a fluid does not exert a force on the fluid, is referred to as 'd'Alembert's paradox'.

If the particle is accelerating, there is a net force exerted on the particle by the fluid. This is calculated by integrating the contribution to the pressure due to acceleration over the surface of the sphere.

$$\begin{aligned} F_i &= \rho \int_{S_{\text{sphere}}} dS n_i \left(\frac{R^3 x_j}{2r^3} \frac{dU_j}{dt} \right) \Bigg|_{r=R} \\ &= \frac{\rho}{2R} \frac{dU_j}{dt} \int_{S_{\text{sphere}}} dS x_i x_j \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{2\pi R^3 \rho}{3}\right) \frac{dU_i}{dt} \\
&= M_b \frac{dU_i}{dt}
\end{aligned} \tag{1.21}
$$

Here, the outward unit normal to the sphere is given by $n_i = (x_i/r)$, and we have used the identity

$$
\int_{S_{\text{sphere}}} dS x_i x_j = \frac{4\pi R^4 \delta_{ij}}{3} \tag{1.22}
$$

to obtain the third step in equation 1.22. Not surprisingly, the force requried to accelerate the particle through the fluid is the product of the added mass and the acceleration.

1.2.2 General three-dimensional potential flows:

Some general results can be obtained for the form of the velocity potential for an object of arbitrary shape translating through a fluid in potential flow. The solution for the velocity potential is, in general, more complicated than the solution 1.6 for the motion of a sphere. However, the leading contribution to the potential is still the dipole contribution which decays proportional to $(1/r^2)$ in three dimensions. There is no source term proportional to $(1/r)$ due to the incompressibility condition and the constant volume of the object. However, higher order terms which decay faster than $(1/r^2)$ could be present in the solution.

Equation 1.18 for the pressure is valid for an object of arbitrary shape, since no assumption was made regarding the specific form of the velocity potential while deriving the pressure.

It is possible to show that the force required for the steady motion of an object of arbitrary shape through the fluid,

$$
\begin{aligned}
F_i &= \int_{S_{\text{object}}} dS p n_i \\
&= \rho \int_{S_{\text{object}}} dS n_i \left(\frac{u_j^2}{2} - U_j u_j\right)
\end{aligned} \tag{1.1}
$$

is equal to zero. Consider the volume integral

$$
\int_V dV \frac{\partial}{\partial x_i} \left(\frac{u_j^2}{2} - U_j u_j\right) \tag{1.2}
$$

evaluated over the volume of the fluid surrounding the fluid. Using the divergence theorem, this volume integral can be written as,

$$
\int_V dV \frac{\partial}{\partial x_i} \left(\frac{u_j^2}{2} - U_j u_j\right) = \int_{S_\infty} dS n_i \left(\frac{u_j^2}{2} - U_j u_j\right) - \int_{S_{\text{object}}} dS n_i \left(\frac{u_j^2}{2} - U_j u_j\right)
$$

$$\int_{S_\infty} dSn_i\left(\frac{u_j^2}{2} - U_j u_j\right) - F_i \tag{1.3}$$

The volume integral on the left side of equation 1.3 can further be simplified as follows,

$$\begin{aligned}\int_V dV\frac{\partial}{\partial x_i}\left(\frac{u_j^2}{2} - U_j u_j\right) &= \int_V dV(u_j - U_j)\frac{\partial u_j}{\partial x_i}\\ &= \int_V dV(u_j - U_j)\frac{\partial u_i}{\partial x_j}\\ &= \int_V dV\frac{\partial}{\partial x_j}((u_j - U_j)u_i)\end{aligned} \tag{1.4}$$

In the second step of the above equation, we have used the fact that the stress tensor is symmetric for an irrotational flow, so that $(\partial u_i/\partial x_j) = (\partial u_j/\partial x_i)$. In the final step, we have used the incompressibility condition $(\partial u_j/\partial x_j) = 0$. The volume integral in equation 1.4 can further be converted into a surface integral by first using the incompressibility condition

$$\int_V dV\frac{\partial}{\partial x_j}((u_j - U_j)u_i) = \int_{S_\infty} dSn_j(u_j - U_j)u_i - \int_{S_{\text{object}}} dSn_j(u_j - U_j)u_i \tag{1.5}$$

In the final step of the above equation, we have used the normal velocity boundary condition, $u_j n_j = U_j n_j$ on the surface of the object, so that the integral over the surface of the object is zero. Equating the right sides of equation 1.3 and 1.6, the force on the object can be expressed as,

$$F_i = \int_{S_\infty} dS\left(n_i\left(\frac{u_j^2}{2} - U_j u_j\right) - n_j(u_j - U_j)u_i\right) \tag{1.6}$$

The right side of the above equation contains terms that are linear or quadratic in the velocity, and the integral is evaluated at a large distance from the object in the limit $r \to \infty$. In this limit, the velocity decreases proportional to $(1/r^3)$, whereas the surface area increases proportional to r^2. Therefore, both the integrals on the right side are zero in the limit $r \to \infty$. This shows that the force on an object of arbitrary shape is zero in a potential flow.

1.3 Two-dimensional potential flows:

The analysis of potential flows in two dimensions is considerably simplified by the use of complex functions. The simplification is due to two important properties of complex functions.

1. If a complex function $F(z)$ of a complex variable z is *analytic*, i. e. the change in the function $DeltaF$ due to a small change in the variable Δz

can be expressed as

$$\Delta F = \frac{dF}{dz}\Delta z, \tag{1.1}$$

then the real and imaginary parts of the complex function satisfy the Laplace equation in the complex plane. Since the velocity potential is also a solution of the Laplace equation, we can associate the real parts of all complex functions to the velocity potential. Thus, there is a potential flow field associated with all complex functions subject to the normal velocity boundary condition at the surfaces.

2. A transformation, called a conformal mapping, can be effected from the independent complex variable z to some other variable z' in such a way that the function $F(z')$ also satisfies the Laplace equation in the z' plane. In this way, it is possible to map the flow in a complicated domain onto the flow in a much simpler domain, and solve the Laplace equation in the simpler domain.

Both the above properties, and their application to two-dimensional complex flows, will be explained in this section.

First, we will show that if the if a complex function is analytic, then the real and imaginary parts satisfy the Laplace equation. Consider a complex function $F(z)$ which is a function of the independent variable $z = x + \imath y$, where $\imath = \sqrt{-1}$. The function can be written as the sum of its real and imaginary parts,

$$F(z) = \phi(x, y) + \imath\psi(x, y) \tag{1.2}$$

The variation in the function F when the independent variable z is displaced by $\Delta z = \Delta x + \imath\Delta y$ is,

$$\begin{aligned} F(z+\Delta z) - F(z) &= \left(\frac{\partial\phi}{\partial x} + \imath\frac{\partial\psi}{\partial x}\right)\Delta x + \left(\frac{\partial\phi}{\partial y} + \imath\frac{\partial\psi}{\partial y}\right)\Delta y \\ &= \left(\frac{\partial\phi}{\partial x} + \imath\frac{\partial\psi}{\partial x}\right)\Delta x + \left(-\imath\frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial y}\right)\imath\Delta y \end{aligned} \tag{1.3}$$

The function is *analytic* if the variation ΔF is proportional to Δz in the limit $\Delta z \to 0$, or if ΔF can be written as,

$$\Delta F = \frac{dF}{dz}\Delta z \tag{1.4}$$

Therefore, for an analytic function, the real and imaginary parts of the coefficients of Δx and $\imath\Delta y$ in equation 1.3 are equal, and the functions ϕ and ψ have the relations,

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \tag{1.5}$$

$$\frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y} \tag{1.6}$$

The above conditions are referred to as the 'Cauchy Reimann' conditions for the analyticity of a complex function. If we sum the partial derivative of equation 1.5 with respect to x and the partial derivative of equation 1.6 with respect to y, we find that the function $\phi(x,y)$ satisfies the Laplace equation in two dimensions,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{1.7}$$

Similarly, if we take the difference of the partial derivative of equation 1.5 with respect to y and equation 1.6 with respect to x, we find that the function ψ also satisfies the Laplace equation in two dimensions,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{1.8}$$

Therefore, if a complex function $F(z)$ is analytic, then its real and imaginary parts satisfy the Laplace equation.

Since the real part $\phi(x,y)$ of any complex function $F(z)$ is a solution of the Laplace equation 1.7, it is also a valid solution for the velocity potential in a two dimensional domain provided it satisfies the normal velocity boundary conditions at the prescribed boundaries. Thus, every complex function has a two dimensional velocity field associated with it that satisfies the potential flow equations. The velocities in the x and y direction are then given by,

$$\begin{aligned} u_x &= \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \\ u_y &= \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \end{aligned} \tag{1.9}$$

where the Cauchy-Reimann conditions have been used to express derivatives of the potential ϕ in terms of derivatives of the imaginary part ψ. It is clear, from the relations 1.9 between u_x, u_y and ψ, that the imaginary part ψ is the stream function for the flow. Thus, the real and imaginary parts of the complex potential are the velocity potential and the stream function respectively. The function $F(z)$ is often referred to as the 'complex potential'. The derivative of $F(z)$ with respect to z can be written as,

$$\begin{aligned} W(z) &= \frac{dF}{dz} \\ &= \frac{\partial \phi}{\partial x} + \imath \frac{\partial \psi}{\partial x} \\ &= u_x - \imath u_y \end{aligned} \tag{1.10}$$

It is also convenient to express the function $W(z)$ in terms of the components (u_r, u_θ) in a polar co-ordinate system. The relation between the velocities in a Cartesian and polar co-ordinate systems are,

$$\begin{aligned} u_x &= u_r \cos(\theta) - u_\theta \sin(\theta) \\ u_y &= u_r \sin(\theta) + u_\theta \cos(\theta) \end{aligned} \tag{1.11}$$

Using these it can easily be verified that the complex potential is,

$$W(z) = (u_r - \imath u_\theta)\exp(-\imath\theta) \tag{1.12}$$

Since we have shown that there are potential flow velocity fields associated with any analytic complex function, we can examine the potential flow fields due to some fundamental forms of the function $F(z)$. We first consider the form,

$$F(z) = Az^m \tag{1.13}$$

where A is assumed to be real without loss of generality. The derivative of $F(z)$ with respect to z is,

$$\begin{aligned} W(z) &= \frac{dF}{dz} \\ &= mAz^{m-1} \\ &= mAr^{m-1}\exp(\imath(m-1)\theta) \end{aligned} \tag{1.14}$$

Comparing the form of $W(z)$ in equation 1.14 with that in equation 1.12, we find that the velocities u_r and u_θ in a polar co-ordinate system are,

$$\begin{aligned} u_r &= Ar^{m-1}\cos(m\theta) \\ u_\theta &= Ar^{m-1}\sin(m\theta) \end{aligned} \tag{1.15}$$

Equations 1.15 represent a flow that, for positive A, is directed radially outward for $\theta = 0$, radially inward for $\theta = (\pi/m)$, and along the θ co-ordinate for $\theta = (\pi/2m)$, as shown in figure **??**. Since the normal velocity is zero along the lines $\theta = 0$ and $\theta = (\pi/m)$, the velocity field 1.15 represents the flow in a corner of subtended angle (π/m). Several special cases can be considered for specific values of m. For $m = 1$, the subtended angle is π, and the flow is a steady linear flow with velocity A in the x direction, as shown in figure **??**. The flow for $m = 2$ is the stagnation point flow in a corner of angle $(\pi/2)$, as shown in figure **??**. The flow for $m > 2$ is the flow in a corner with a subtended acute angle, while the flow for $1 < m < 2$ is the flow in a corner with a subtended obtuse angle, as shown in figure **??**.

Next, we consider the form

$$F(z) = \frac{m}{2\pi}\log(z) \tag{1.16}$$

where m is a real constant. The derivative of $F(z)$ with respect to z is,

$$\begin{aligned} W(z) &= \frac{m}{2\pi z} \\ &= \frac{m}{2\pi r}\exp(-\imath\theta) \end{aligned} \tag{1.17}$$

Comparing equation 1.17 with 1.12, we find that the components of the velocity field are,

$$\begin{aligned} u_r &= \frac{m}{2\pi r} \\ u_\theta &= 0 \end{aligned} \tag{1.18}$$

This is the flow from a point source of fluid in two dimensions, from which the total volume of fluid generated (per unit length in the third dimension) Q at any radius R is equal to m, and is independent of radius.

A logarithmic function with an imaginary coefficient represents circulation about the origin. Consider the form,

$$F(z) = \frac{\imath\Gamma}{2\pi}\log(z) \tag{1.19}$$

where Γ is real. The derivative of $F(z)$ is,

$$\begin{aligned} W(z) &= -\frac{\imath\Gamma}{2\pi z} \\ &= -\frac{\imath\Gamma}{2\pi r}\exp(-\imath\theta) \end{aligned} \tag{1.20}$$

Comparing equation 1.20 with equation 1.12, we find the components of the velocity,

$$\begin{aligned} u_r &= 0 \\ u_\theta &= \frac{\Gamma}{2\pi r} \end{aligned} \tag{1.21}$$

Thus, the flow due to the potential 1.21 is a circulating flow around the origin. For this flow, the circulation is the integral of the tangential velocity along a closed curve around the origin, is equal to Γ.

$$\begin{aligned} \text{Circulation} &= \int_0^{2\pi} r d\theta u_\theta \\ &= \Gamma \end{aligned} \tag{1.22}$$

1.3.1 Flow around a cylinder

The motion around a cylinder translating with a velocity U_x in the x direction in a fluid which is at rest at a large distance from the cylinder is expressed by a complex potential of the form,

$$F(z) = -\frac{U_x R^2}{z} - \frac{\imath\Gamma}{2\pi}\log(z) \tag{1.1}$$

The velocity fields can be inferred from the derivative of this complex potential with respect to z,

$$\begin{aligned} W(z) &= \frac{dF}{dz} \\ &= \frac{U_x R^2}{z^2} - \frac{\imath\Gamma}{2\pi z} \\ &= \left(\frac{U_x R^2}{r^2}\exp(-\imath\theta) - \frac{\imath\Gamma}{2\pi r}\right)\exp(-\imath\theta) \end{aligned} \tag{1.2}$$

The radial and polar components of the velocity can be inferred using equation 1.12,

$$\begin{aligned} u_r &= \frac{U_x R^2}{r^2}\cos(\theta) \\ u_\theta &= \frac{U_x R^2}{r^2}\sin(\theta) + \frac{\Gamma}{2\pi r} \end{aligned} \tag{1.3}$$

The above velocity field satisfies the condition $u_i n_i = U_i n_i$ at the surface of a cylinder with radius R, since the normal velocity $u_i n_i$, which is the radial velocity u_r in a polar co-ordinate system, is equal to $U_i n_i = U_x n_x = U_x \cos(\theta)$, where the component n_x of the unit normal to the surface is equal to $\cos(\theta)$.

The force on the cylinder can be determined from the equation 1.17 for the force on an object moving with velocity U_i through the fluid,

$$\begin{aligned} F_i &= \rho \int_{S_{\text{object}}} dS n_i \left(\frac{u_j^2}{2} - U_j u_j \right) \\ &= \rho \int_0^{2\pi} (R d\theta)\, n_i \left(\frac{u_\theta^2}{2} - U_x (u_r \cos(\theta) - u_\theta \sin(\theta) \right) \Bigg|_{r=R} \end{aligned} \tag{1.4}$$

The 'drag' force F_x, which is in the direction of the velocity, can be determined using $n_x = \cos(\theta)$ in equation 1.4,

$$\begin{aligned} F_x &= \rho \int_0^{2\pi} (R d\theta) \cos(\theta) \left(\frac{1}{2} \left(U_x \sin(\theta) + \frac{\Gamma}{2\pi R} \right)^2 + U_x \left(u_r \cos(\theta) + u_\theta \sin(\theta) \right) \right) \\ &= \rho \int_0^{2\pi} (R d\theta) \cos(\theta) \left(\frac{1}{2} \left(U_x \sin(\theta) + \frac{\Gamma}{2\pi R} \right)^2 + U_x^2 \left(\cos(\theta)^2 - \sin(\theta)^2 \right) \right) \\ &= 0 \end{aligned} \tag{1.5}$$

As in three dimensions, the drag force due to the potential flow in two dimensions is also equal to zero. The 'lift' force perpendicular to the direction of gravity can be determined using $n_y = \sin(\theta)$ in equation 1.4,

$$\begin{aligned} F_x &= \rho \int_0^{2\pi} (R d\theta) \sin(\theta) \left(\frac{1}{2} \left(U_x \sin(\theta) + \frac{\Gamma}{2\pi R} \right)^2 + U_x^2 \left(\cos(\theta)^2 - \sin(\theta)^2 \right) \right) \\ &= \rho U_x \Gamma \end{aligned} \tag{1.6}$$

1.4 Force on a two-dimensional object of arbitrary shape:

The net force exerted on a two-dimensional object in potential flow can be calculated in a manner similar to that for a three-dimensional flow. A procedure identical to that in section **??** will result in the equation 1.6 for a two dimensional object,

$$F_i = \int_{S_\infty} dS \left(n_i \left(\frac{u_j^2}{2} - U_j u_j \right) - n_j (u_j - U_j) u_i \right) \tag{1.1}$$

In two dimensions, the surface area (per unit length in the direction perpendicular to the plane of flow) of a surface with radius r increases proportional to r in two dimensions. Therefore, the force is non-zero only if the integrand decreases proportional to r in the limit $r \to \infty$. In two dimensions, the most slowly decaying velocity field is due to a line source of fluid equation 1.18, and due to a line vortex equation 1.21, both of which decay proportional to $(1/r)$ in the limit $r \to \infty$. In the absence of a source of fluid within the object, a non-zero contribution to the force can be caused only by a line vortex with velocity $u_\theta = (\Gamma/2\pi r)$ where Γ is the circulation. co-ordinates, or $u_i = \epsilon_{ijk}\Gamma_j n_k/(2\pi r)$ in Cartesian co-ordinates, where Γ is the circulation, and $\Gamma_i = \Gamma\delta_{iz}$. In equation 1.1, there is no contribution due to the terms proportional to u_j^2 and $u_i u_j$, since these decay proportional to $(1/r)$ in the limit $r \to \infty$. Therefore, it is sufficient to consider the components of the integrand of equation 1.1 linear in the velocity u_i,

$$F_i = \rho \int_{S_\infty} dS U_j (n_j u_i - n_i u_j) \tag{1.2}$$

If we choose the direction of the mean velocity to be along the x direction without loss of generality, it is clear that the 'drag' force along the x direction is zero from equation 1.2. However, the 'lift' force along the y direction is not zero, and can be calculated by writing the velocity $u_\theta = (\Gamma/2\pi r)$ (equation 1.2) in terms of the Cartesian components, $u_x = (-\Gamma \sin(\theta)/(2\pi r))$ and $u_y = (\Gamma \cos(\theta)/(2\pi r))$,

$$\begin{aligned} F_y &= \rho \int_{S_\infty} dS U_x (n_x u_y - n_y u_x) \\ &= \rho \int_{S_\infty} dS U_x \Gamma/(2\pi r) \\ &= \rho U \Gamma \end{aligned} \tag{1.3}$$

Problems:

1. Find the added mass per unit length of an infinite cylinder in potential flow, using a procedure identical to that for a three-dimensional object derived here.

2. Determine the total kinetic energy due to a moving sphere under potential flow conditions. From this, calculate the added mass. Also, find the total kinetic energy of the fluid when a sphere moves in the limit of zero Reynolds number. How do the two compare?

3. Determine the energy dissipation due to the motion of a sphere at high Reynolds number, assuming that the fluid velocity field is given by the potential flow solution. Using this, find the drag force on the sphere. How does it compare to the drag force on a sphere at zero Reynolds number? The rate of dissipation of energy is given by,

$$D = \int dV \mu (\nabla u : \nabla u)$$

4. Consider a bubble with internal pressure p_b expanding in a fluid in which the pressure a large distance from the bubble is p_0. The radius of the bubble $R(t)$ is a function of time as it expands.

   (a) Solve the potential flow equations to determine the fluid velocity field due to the expanding bubble.

   (b) Determine the pressure at the surface of the bubble.

   (c) Form a pressure balance condition, find the equation for the evolution of the bubble radius.

5. Consider two line sources of strength $-m$ and m separated by a distance d along the x axis, in a fluid flowing with a constant velocity U in the x direction. What is the equation for the shape of the object which is equivalent to these two sources? Show that in the limit $d \to 0$ and (md) finite, the object assumes the shape of a cylinder, and find its radius.

6. Consider the transform from the z to the z' plane given by

$$z = z' + \frac{a^2}{z'}$$

   (a) Determine the relations between the coordinates (x', y') and (x, y).

   (b) Consider a circle of radius a in the z' plane. What is the transformed shape in the z plane? If we consider the flow past the cylinder in the z' plane

$$F(z') = z' + \frac{a^2}{z'}$$

   in the z' plane, what is the equivalent flow in the z plane?

   (c) Consider a circle of radius $b > a$ in the z' plane. What is the transformed shape in the z plane? If we consider the flow past the cylinder in the z' plane

$$F(z') = z' + \frac{b^2}{z'}$$

   in the z' plane, what is the equivalent flow in the z plane?

7. A three dimensional irregular body is moving with a velocity U_1 in the x_1 direction near a wall which is perpendicular to the x_2 axis as shown in the figure. The Reynolds number is large so that the potential flow equations are applicable. The wall is impermeable so that the fluid flow at the wall is tangential to it. Find the force on the body in the x_1 and x_2 directions as a function of the velocity of the object U_1 and the fluid velocity at the wall.
(**Hint:** Consider a control volume bounded by the surface of the object S, the surface of the wall S_w and the surface at infinity S_∞. Use methods similar to the derivation of the d'Alembert paradox for a general body in irrotational flow.)

8. Consider the dynamics of waves on the surface of a liquid of wavelength λ, frequency ω and amplitude ξ_0. Find the conditions (at high Reynolds number) under which the $u_j \partial_j u_i$ term in the momentum conservation equation can be neglected compared to the $\partial_t u_i$ term. Under these conditions, find the frequency of the surface waves on the surface of a liquid of infinite depth as a function of the wavelength of the waves. The height of the surface fluctuations is given by the equation:

$$\xi = \xi_0 \exp\left(ikx + i\omega t\right)$$

where $k = (2\pi/\lambda)$ is the wave number, and ω is the frequency of the waves. In addition, the normal velocity at the surface $z = 0$ is given by the time rate of change of the displacement ξ.