Fundamentals of Transport Processes 2

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Course outline and reading material:

	Subject	Lectures	Reading
1.	Introduction.	1-2	
2.	Vectors & Tensors.	3-6	Griffiths 1.
3.	Kinematics.	7-8	Batchelor 2
4.	Navier-Stokes equations.	9-14	Batchelor 3.1-3.4
5.	Viscous flow.	15-24	Batchelor 1.9, 4.10-4.11, Leal 4A-F,5A-B
6.	Potential flow.	25-30	Panton 18, Batchelor 6.
7.	Boundary layer theory.	30-37	Leal 11A-E,12A-B
8.	Turbulence	38-40	Tennekes & Lumley 1,2,5.1-5.2

1. D. J. Griffiths, An Introduction to Electrodynamics, Prentice Hall International, 1994.

2. G. K. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press, 1967.

3. L. G. Leal, Advanced Transport Phenomena, Cambridge University Press, 2007.

4. R. L. Panton, Incompressible flow, John Wiley & Sons, New York, 1984.

5. H. Tennekes and J. L. Lumley, A first course in turbulence, The MIT Press, 1972.

Exercises

1 Vector and tensor analysis:

1. Verify if the following expressions for tensors are correct, and determine their order.

   (a) $A_{ijkl}B_{mk}$ (b) $L_{ijm}K_{imn}M_{kmn}$
   (c) $S_{ijil}H_{jml}$ (d) $X_{ij}Y_{il}Z_{jl}$

2. Does the order of appearance of the components make a difference in the following expressions

   (a) $\epsilon_{ijk}a_j b_k$ and $b_k a_j \epsilon_{ijk}$

   (b) $\rho\partial_i v_j$ and $v_j\partial_j\rho$

   (c) $a_{ij}b_k$ and $a_{ik}b_j$

3. The stresses acting on the faces of a cube of unit length in all three directions are as follows:

$$T_{ij} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix}, \tag{1}$$

   where i denotes the direction of the force and j denotes the direction of normal to the area. Find out the force acting on the faces of the cube, and the torque on the cube in the three directions. For calculating the torque, place the origin of the coordinate system at the center of the cube. The torque is the cross product of the force acting on the cube and the displacement from the center of the cube.

   If it is required that the net torque in any direction should be zero, what is the condition on T_{ij}?

4. Show that:

   (a) $\nabla \times \nabla\phi = 0$ Interpret this in terms of contours of the scalar function ϕ.

   (b) $\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla\cdot\mathbf{u}) - \nabla^2\mathbf{u}$. (Hint: Prove that $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ and use this result.)

(c) $S_{ij}A_{ij} = 0$ where S_{ij} and A_{ij} are a symmetric and antisymmetric matrix respectively.

(d) An antisymmetric tensor A_{ij} may be written as $A_{ij} = \epsilon_{ijk}\omega_k$ where $\omega_k = (1/2)\epsilon_{klm}A_{lm}$.

5. Derive an expression for $\nabla^2\phi$ in terms of the coordinates x_a, x_b and x_c and the scale factors h_a, h_b and h_c for a curvilinear coordinate system.

6. Derive expressions for $\mathbf{e}_i$, h_i and $(\partial \mathbf{e}_i/\partial x_j)$ for a cylindrical coordinate system.

7. Let $f(r)$ be any scalar function of the magnitude $r = |\mathbf{r}|$ of the position vector $\mathbf{r}$ relative to the center of a sphere.

   (a) Evaluate the integral:

$$\int_V dV\ f(r)\,\mathbf{r}\mathbf{r} \tag{2}$$

   over the volume of a spherical sector with angle θ_0 and axis in the $\mathbf{a}$ direction. (Hint: After integrating, what vectors or tensors can the result depend on?)

   (b) What is the result when the sector is the entire sphere?

   (Express your result in terms of integrals over the radius r).

8. Consider a two - dimensional coordinate system given by:

$$x = \cosh{(\xi)}\cos{(\eta)} \quad y = \sinh{(\xi)}\sin{(\eta)} \tag{3}$$

   where

$$\cosh{(\xi)} = \frac{\exp{(\xi)} + \exp{(-\xi)}}{2} \quad \sinh{(\xi)} = \frac{\exp{(\xi)} - \exp{(-\xi)}}{2} \tag{4}$$

   and

$$\frac{\partial \cosh{(\xi)}}{\partial \xi} = \sinh{(\xi)} \quad \frac{\partial \sinh{(\xi)}}{\partial \xi} = \cosh{(\xi)} \tag{5}$$

   (a) Derive an expression for $\mathbf{e}_\xi$ and $\mathbf{e}_\eta$ in terms of $\mathbf{e}_x$, $\mathbf{e}_y$, ξ and η. **Note:** In order to determine the unit vectors, it is not necessary to invert the expressions in equation 162 to determine $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$.

(b) Is the coordinate system an orthogonal one?

(c) Determine the scale factors h_ξ and h_η.

9. Derive an expression for $\nabla^2\phi$ and $\nabla\times\mathbf{A}$ in terms of the coordinates x_a, x_b and x_c and the scale factors h_a, h_b and h_c for a curvilinear coordinate system. The final expressions you get should be,

$$\nabla^2\phi = \frac{1}{h_a h_b h_c}\left(\frac{\partial}{\partial x_a}\left(\frac{h_b h_c}{h_a}\frac{\partial\phi}{\partial x_a}\right) + \frac{\partial}{\partial x_b}\left(\frac{h_a h_c}{h_b}\frac{\partial\phi}{\partial x_b}\right) + \frac{\partial}{\partial x_c}\left(\frac{h_a h_b}{h_c}\frac{\partial\phi}{\partial x_c}\right)\phi\right) \tag{6}$$

while for the curl, the final expression is,

$$\nabla\times\mathbf{A} = \begin{pmatrix} h_a\mathbf{e}_a & h_b\mathbf{e}_b & h_c\mathbf{e}_c \\ \frac{\partial}{\partial x_a} & \frac{\partial}{\partial x_b} & \frac{\partial}{\partial x_c} \\ h_a A_a & h_b A_b & h_c A_c \end{pmatrix} \tag{7}$$

2 Kinematics:

1. Consider a linear velocity profile

$$v_x = \gamma y \tag{8}$$

in a two dimensional $x-y$ coordinate system. Separate the deformation into its fundamental components, and sketch these for the flow under consideration.

2. Consider the parabolic flow of a fluid in a tube of radius R, with the velocity given in cylindrical coordinates by:

$$v_z = V\left(1 - \frac{r^2}{R^2}\right) \tag{9}$$

Separate the rate of deformation tensor into its elementary components.

3. The velocity profile of a fluid in cylindrical coordinates (r,ϕ,z) is given by

$$v_\phi = \frac{\Omega}{r} \tag{10}$$

where Ω is a constant, and the velocity is independent of the z coordinate. This flow appears to be rotational.

- Calculate the symmetric and antisymmetric parts of the deformation tensor. Do this in Cartesian coordinates, and in cylindrical coordinates. Are they different? Why?
- Calculate the vorticity $\omega = \nabla \times \mathbf{v}$ for this flow. Can you explain the results?
- What is the vorticity at the origin?

4. A 'four roll mill', which is used to carry out experiments on the effect of deformation on complex fluids, consists of four rollers rotating as shown in figure 1. The velocity field at the center point between these four rollers can be approximated by

$$
\begin{aligned}
u_x &= L\alpha(\Omega_{up} - \Omega_{down}) - \beta y(\Omega_{right} + \Omega_{left}) \\
u_y &= -L\alpha(\Omega_{right} - \Omega_{left}) + \beta x(\Omega_{up} + \Omega_{down})
\end{aligned}
$$

where L and the angular velocities are shown in the figure, and α and β are constants. Consider the steady flow of a Newtonian fluid in the four roll mill.

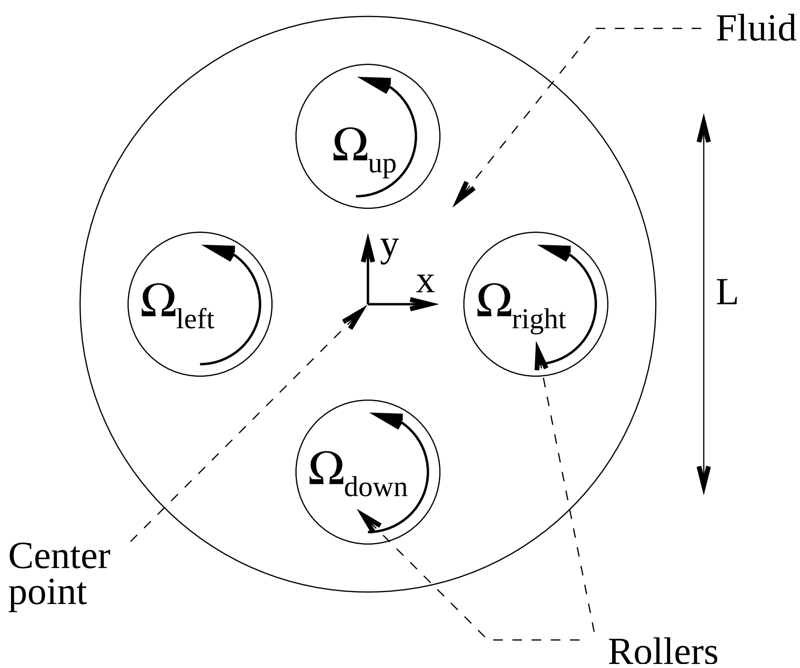

Figure 1: Four roll mill.

(a) In practical applications, it is necessary to have a stagnation point at the center $x = 0$ and $y = 0$ shown in the figure, where the mean velocity is zero. What is the condition for zero mean velocity at this point?

(b) Consider a flow with zero mean velocity at $x = 0$ and $y = 0$. What is the rate of deformation tensor at this point?

(c) Under what conditions is the flow purely extensional with no rotational component?

(d) Under what conditions it the flow purely rotational with no extensional component?

(e) Under what conditions is the flow a linear shear flow near the center, in which the velocity is a linear function of position?

3 Conservation equations:

1. Show that the rate of dissipation of energy, given by

$$D = (S_{ij} - (\delta_{ij}/3) S_{kk}) S_{ij}) \tag{11}$$

is always positive, where S_{ij} is the symmetric traceless part of the rate of deformation tensor.
Hint: Try to express this as the sum of squares.

4 Viscous flows:

1. Two spheres of equal radius are falling due to gravity in a viscous fluid, and the line joining their centres is at an angle θ to the vertical as shown in Figure 1. The velocities of the two spheres, $\mathbf{u}_1$ and $\mathbf{u}_2$, can be separated into the velocity of the centre of mass $\mathbf{v} = (\mathbf{u}_1 + \mathbf{u}_2)/2$ and the velocity difference between the spheres $\mathbf{w} = (\mathbf{u}_1 - \mathbf{u}_2)$. These can further be separated into the components along the line joining the centres, $\mathbf{v}_\parallel$ and $\mathbf{w}_\parallel$, and the velocities perpendicular to the line of centres, $\mathbf{v}_\perp$ and $\mathbf{w}_\perp$. Using Stokes flow reversibility and symmetry, determine which of these four components can be non - zero, and which are identically zero.

2. G. I. Taylor showed that the sedimentation velocity $\mathbf{U}_{\parallel}$ of a needle like object (or slender body) when its axis is vertical (parallel to gravity) is twice the sedimentation velocity $\mathbf{U}_{\perp}$ when the axis is perpendicular to gravity, i. e., $\mathbf{U}_{\parallel} = 2\mathbf{U}_{\perp}$. Using this information, find a general equation that relates the sedimentation velocity U_i and orientation p_i of the slender body to $U_{\parallel}$ and g_i, the unit vector in the direction of gravity. Assume zero Reynolds number flow.

3. Determine the fluid flow field and the stress acting on a particle of radius a placed in an extensional strain field $u_i = G_{ij}x_j$ ($G_{ij} = G_{ji}$) at low Reynolds number where the fluid and particle inertia can be neglected. The particle is placed at the origin of the coordinate system, and the fluid velocity field has the undisturbed value $u_i = G_{ij}x_j$ for $r \to \infty$. Find the fluid velocity and pressure fields around the particle. Find the integral:
$$\int_A dA T_{il} n_l x_j \tag{12}$$
over the surface of the sphere, where n_l is the outward unit normal.

4. A sphere is rotating with an angular velocity Ω_k in a fluid that is at rest at infinity, and the Reynolds number based on the angular velocity and radius of the sphere, $Re \equiv \rho\Omega a^2/\mu$ is small. The velocity at the surface of the particle is:
$$u_i|_{r=a} = \epsilon_{ikl}\Omega_k x_l \tag{13}$$
and the velocity and pressure fields decay far from the sphere.
   (a) Find the most general form for the velocity and pressure fields at zero Reynolds number. Note that the velocity and pressure are real vectors, whereas Ω_k is a pseudo vector.
   (b) Use the incompressibility condition and the condition on the velocity at the surface of the sphere to determine the constants in your expression for the fluid velocity.
   (c) Determine the force per unit area f_i exerted on the surface of the sphere. Note that the force is a function of position on the surface of the sphere.

(d) The torque on the particle is

$$L_i = \int_A dA \epsilon_{ijk} f_j n_k \tag{14}$$

where f_j is the force per unit area exerted on the sphere at position x_n. Find the torque.

5. Find the force necessary to move a disk of radius a towards a plane solid boundary with a velocity U when the gap between the disk and the boundary h is small, $h = \epsilon a$ where $\epsilon \ll 1$.

6. A two dimensional cylinder of radius a is moving with a velocity U along the center of a two dimensional channel of width $2a(1+\epsilon)$, where $\epsilon \ll 1$ as shown in Figure 1. The end of the slot is closed so that the fluid displaced by the cylinder has to escape through the narrow gap between the cylinder and the channel.

   (a) Choose a coordinate system for analysing the flow in the gap between the cylinder and the channel wall. Write the Navier Stokes equations for the flow in the gap, and scale the equations appropriately. Under what conditions can the inertial terms in the conservation equation be neglected?

   (b) What are the boundary conditions required to solve the problem? What is the other condition due to the requirement that the volume displaced by the cylinder has to flow through the gap?

   (c) Determine the velocity and pressure fields in the gap when the inertial terms in the conservation equation can be neglected.

   (d) Calculate the forces on the cylinder along the center line of the channel and perpendicular to it.

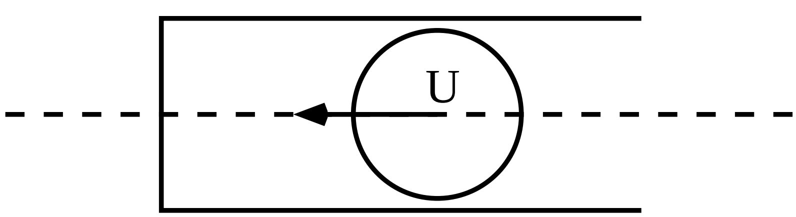

Figure 2: Two dimensional cylinder moving in a slot.

5 Potential flow:

1. Find the added mass per unit length of an infinite cylinder in potential flow, using a procedure identical to that for a three-dimensional object derived in class.

2. Consider two line sources of strength $-m$ and m separated by a distance d along the x axis, in a fluid flowing with a constant velocity U in the x direction. What is the equation for the shape of the object which is equivalent to these two sources? Show that in the limit $d \to 0$ and (md) finite, the object assumes the shape of a cylinder, and find its radius.

3. (a) Find the total kinetic energy of the fluid flow due to a sphere of radius a moving with a velocity U_i in potential flow.

$$KE = \int_V dV \frac{1}{2} \rho u_i^2 \tag{15}$$

   where the integral is carried out over the volume of the fluid.

   (b) What would you get if you attempt to determine the kinetic energy for the flow at zero Reynolds number? How would you estimate the magnitude of the kinetic energy in this case?

   (c) Which would you expect to be greater, and why?

4. A fluid is contained in the annulus between two concentric cylinders of radius R_1 and R_2 and of infinite height, and the cylinders are rotated with angular velocity Ω_1 and Ω_2 respectively.

   (a) What is the condition for the flow in the gap between the two cylinders to be irrotational? Find the velocity profile for the irrotational flow in the annulus.

   (b) Find the equation of the surface of the fluid if the viscous effects are neglected. Sketch the surface.

5. Consider the transform from the z to the z' plane given by

$$z = z' + \frac{a^2}{z'}$$

(a) Determine the relations between the coordinates (x', y') and (x, y).

(b) Consider a circle of radius a in the z' plane. What is the transformed shape in the z plane? If we consider the flow past the cylinder in the z' plane

$$F(z') = z' + \frac{a^2}{z'}$$

in the z' plane, what is the equivalent flow in the z plane?

(c) Consider a circle of radius $b > a$ in the z' plane. What is the transformed shape in the z plane? If we consider the flow past the cylinder in the z' plane

$$F(z') = z' + \frac{b^2}{z'}$$

in the z' plane, what is the equivalent flow in the z plane?

6. Consider a bubble with internal pressure p_b expanding in a fluid in which the pressure a large distance from the bubble is p_0. The radius of the bubble $R(t)$ is a function of time as it expands.

   (a) Solve the potential flow equations to determine the fluid velocity field due to the expanding bubble.

   (b) Determine the pressure at the surface of the bubble.

   (c) Form a pressure balance condition, find the equation for the evolution of the bubble radius.

7. Determine the energy dissipation due to the motion of a sphere at high Reynolds number, assuming that the fluid velocity field is given by the potential flow solution. Using this, find the drag force on the sphere. How does it compare to the drag force on a sphere at zero Reynolds number? The rate of dissipation of energy is given by,

$$D = \int dV \mu (\nabla u : \nabla u)$$

8. A three dimensional irregular body is moving with a velocity U_1 in the x_1 direction near a wall which is perpendicular to the x_2 axis as shown in Figure 2. The Reynolds number is large so that the potential flow

equations are applicable. The wall is impermeable so that the fluid flow at the wall is tangential to it. Find the force on the body in the x_1 and x_2 directions as a function of the velocity of the object U_1 and the fluid velocity at the wall.
(**Hint:** Consider a control volume bounded by the surface of the object S, the surface of the wall S_w and the surface at infinity S_∞. Use methods similar to the derivation of the d'Alembert paradox for a general body in irrotational flow.

9. Consider the dynamics of waves on the surface of a liquid of wavelength λ, frequency ω and amplitude ξ_0. Find the conditions (at high Reynolds number) under which the $u_j \partial_j u_i$ term in the momentum conservation equation can be neglected compared to the $\partial_t u_i$ term. Under these conditions, find the frequency of the surface waves on the surface of a liquid of infinite depth as a function of the wavelength of the waves. The height of the surface fluctuations is given by the equation:

$$\xi = \xi_0 \exp{(ikx + i\omega t)}$$

where $k = (2\pi/\lambda)$ is the wave number, and ω is the frequency of the waves. In addition, the normal velocity at the surface $z = 0$ is given by the time rate of change of the displacement ξ.

10. Consider the **extensional flow** around a spherical particle in the figure, where the velocity normal to the sphere surface is zero on the surface of the particle $r = R$, and the velocity is $u_i = G_{ij} x_j$ far from the particle $r \to \infty$, where G_{ij} is a symmetric traceless tensor. Assume the flow satisfies the potential flow equations, where the velocity is defined as the gradient of a potential, $u_i = (\partial \phi / \partial x_i)$, and the potential satisfies the Laplace equation $\nabla^2 \phi = 0$.

    (a) What is the normal velocity condition at the surface of the sphere?

    (b) If the potential far from the surface $r \to \infty$ is given by $\phi = C G_{ij} x_i x_j$, what is the value of the constant C so that the velocity is $u_i = G_{ij} x_j$ for $r \to \infty$?

    (c) Close to the particle surface, there is a disturbance to the potential due to the zero normal velocity condition. This disturbance is linear in the second order tensor G_{ij} as well as linear in the terms

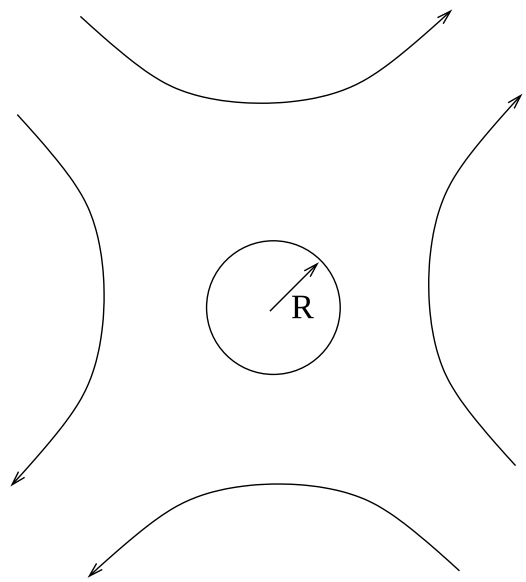

Figure 3: Extensional flow around a sphere.

in the spherical harmonic expansion. What is the form of the disturbance to the potential?

(d) Determine the constants in the expressions for the potential and velocity so that the zero normal velocity conditions are satisfied on the surface.

6 Boundary layer theory:

1. Consider the uniform flow of a fluid past a flat plate of infinite extent in the $x_1 - x_3$ plane, with the edge of the plate at $x_1 = 0$. The Reynolds number based on the fluid velocity and the length of the plate is large. At a large distance from the plate, the fluid has a uniform velocity U_1 in the x_1 direction and U_3 in the x_3 direction. All velocities are independent of the x_3 direction, and the velocities U_1 and U_3 are uniform and independent of position in the outer flow,

   (a) Write down the equations of motion in the x_1, x_2 and x_3 directions. If the thickness of the boundary layer δ is small compared to the length of the plate L, find the leading order terms in the conservation equations.

(b) What is the similarity form of the equation for the momentum equations for the velocity u_1? Make use of the similarity forms for boundary layer flows derived in class. **Do not try to solve the equation.**

(c) Find the solution for the velocity u_3 in terms of the solution for u_1.

(d) If the velocity U_1 in the outer flow is a constant, but the velocity $U_3 = U_{30} + U_{31}x_1$, what is the form of the equation for the velocity u_3? Is a similarity solution possible?

2. Consider the Blasius boundary layer flow of a fluid past a flat plate of length L in the high Reynolds number limit, where x and y are the streamwise and cross-stream directions respectively. The velocity profile is given by the Blasius boundary layer profile. The temperature of the fluid at a large distance from the plate is T_0. For $x < 0$, the temperature of the plate is T_0, while for $x > 0$, the plate is at a higher temperature $T_1 > T_0$. The equation for the temperature field in the fluid is:

$$\frac{\partial T}{\partial t} + \mathbf{u}.\nabla T = \alpha \nabla^2 T$$

where $\mathbf{u}$ is the fluid velocity. The above equation gives a balance between the convective and diffusive transport of heat in the fluid.

(a) Scale the coordinates in the temperature equation, and determine the equivalent Peclet number for this problem.

(b) Assume that the temperature field can be expressed in terms of the similarity variable using a suitable form, and derive a similarity equation for the temperature field, which contains the Prandtl number, the ratio of the Peclet and Reynolds numbers.

(c) If this number is large, one would expect the conduction to be confined to a thin layer near the surface of the plate. Scale the coordinates in the heat balance equation in this case, and determine the boundary layer thickness as a function of the Prandtl number.

(d) What is the scaling of the flux as a function of the Prandtl and Reynolds numbers?

3. Consider free convection of heat from a plate of height H in the y direction (opposite to the diretion of gravity), of infinitesimal width and of infinite length in the direction perpendicular to the page, as shown in figure 4. The temperature of the surface of the plate is T_1, and the temperature far away is T_0. The velocity is zero both at the surface of the plate, and far away from the plate. However, the hot fluid near the plate is lighter, so it rises. The conservation equations are given by the Bousinessq approximation,

$$\nabla.\mathbf{u} = 0$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u}\right) = -\nabla p - \beta(T - T_0)g\mathbf{e}_y + \mu\nabla^2\mathbf{u}$$

$$\mathbf{u}.\nabla T = \alpha\nabla^2 T$$

Here β is the coefficient of thermal expansion, ρ is the density and g is the acceleration due to gravity, and $\mathbf{e}_y$ is the vector in the upward direction.

a) There is no imposed velocity, the velocity and pressure scales are determined from a balance between inertial and buoyancy forces. What are the velocity and pressure scales?

b) Non-dimensionalise the equations using the velocity scale derived above, in such a way that there is a balance between buoyancy and inertial forces. The resulting equations should contain two dimensionless numbers, the Prandtl number $(\mu/\rho\alpha)$ and a second dimensionless number which measures the ratio of viscous and buoyancy forces. Identify this number.

c) When the Prandtl number is large and the ratio of viscous and buoyancy forces is $O(1)$, the heat transfer is confined to a thin boundary layer at the surface of the plate. Assume a boundary layer of thickness δ in the x direction near the plate, while the length scale in the y direction is still H. Obtain the scaled equations.

d) Determine the scaling for u_y in terms of δ by balancing the viscous and buoyancy terms in the y momentum equation, and then determine the scaling for u_x in terms of δ from the continuity equation.

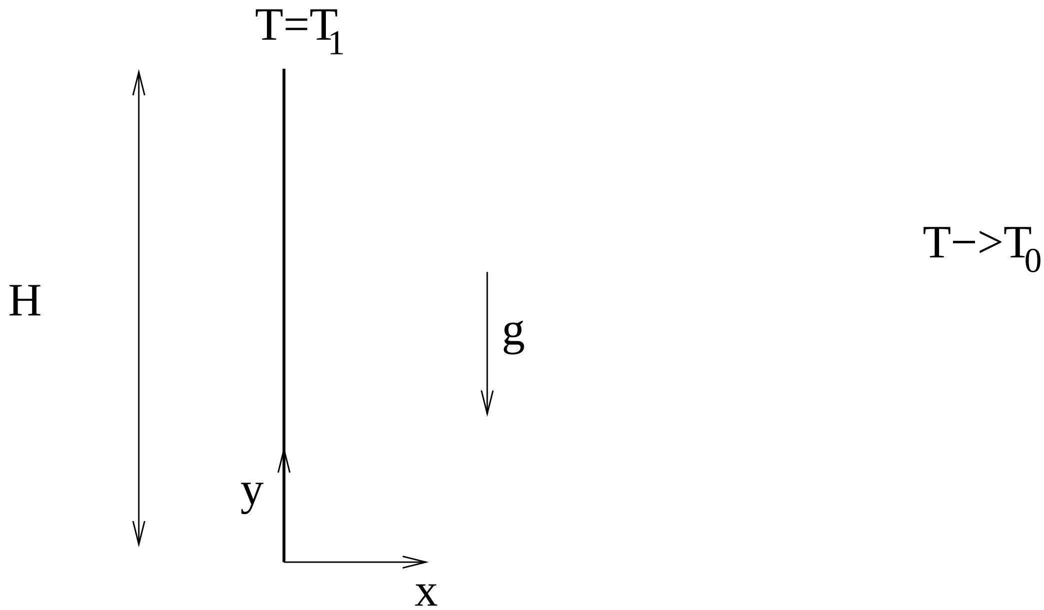

Figure 4: Convetion from a vertical plate.

e) Insert these scalings into the equation for the temperature, in order to obtain δ in terms of the Prandtl number in the limit $Pr \gg 1$.

4. Consider the biaxial extensional flow towards a flat surface shown in the figure. For the outer inviscid flow, the velocity field is given by,

$$\begin{aligned} u_x &= Ax \\ u_y &= -(A+B)y \\ u_z &= Bz \end{aligned}$$

where A and B are constants. At the surface, the no-slip conditions require that $u_x = 0, u_y = 0, u_z = 0$. Obtain a boundary layer solution for the flow near the surface as follows.

(a) What are the simplified equations for the boundary layer flow, where the boundary layer thickness is small compared to x and z?

(b) How would you approximate the pressure gradients in the momentum equations in the boundary layer?

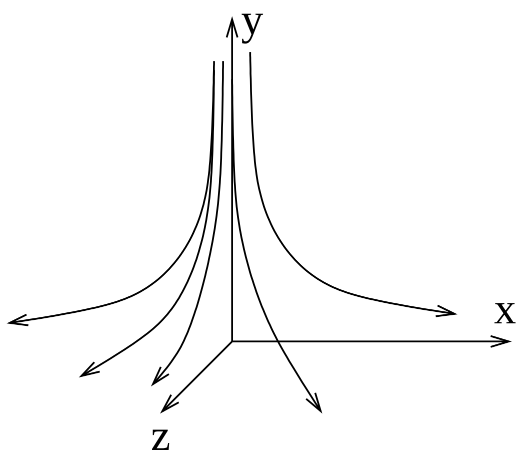

Figure 5: Biaxial extensional flow towards a surface.

(c) Assume forms for the velocity profiles u_x and u_z in the boundary layer as,

$$\begin{aligned} u_x &= Axf'(\eta) \\ u_z &= Azg'(\eta) \end{aligned}$$

where $\eta = (y/\sqrt{\nu/A})$ is the similarity variable, and the prime denotes a derivative with respect to η. From the mass conservation equation, what is the velocity u_y in the boundary layer?

(d) Substitute the similarity forms for the velocity in the boundary layer equations. What are the equations for f and g?

(e) What are the boundary conditions for f and g?

(f) Is it possible to obtain a similarity solution?