The rate of coalescence in a suspension of high Reynolds number, low Weber number bubbles

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The interactions between bubbles of nearly equal size (size ratio less than 1.1 at Re=100, and size ratio less than 1.07 at Re=200) in a suspension of bubbles rising due to gravity were analyzed under high Reynolds number, low Weber number conditions. The potential flow interaction between a pair of bubbles was studied using the equations of motion derived by Kumaran and Koch [Phys. Fluids A **5**, 1123 (1993)]. If the horizontal distance between the bubbles is smaller than a critical value, the bubbles approach each other along the horizontal plane and collide. Previous studies suggest that in a collision, the bubbles could either coalesce or bounce off each other. If it is assumed that the bubbles bounce when they collide, they continue to collide repeatedly. The amplitude of the oscillations decrease due to viscous drag, and ultimately the bubbles rise as a horizontally aligned pair. In this configuration, scaling arguments indicate the bubbles coalesce in finite time. The coalescence frequency in a suspension of bubbles was calculated by averaging over pair interactions. This procedure is valid for $V \ll 18/\mathrm{Re}$, where V is the volume fraction of the bubbles. The coalescence frequency is proportional to $V\,\mathrm{Re}^{-2/5}$ in this limit.

I. INTRODUCTION

In this paper, we study the coalescence of bubbles in a suspension under high Reynolds number, low Weber number conditions. Air bubbles of radius between 0.4 and 0.8 mm rising under gravity in water satisfy these conditions, because the Reynolds number, $\mathrm{Re}=\rho U a/\eta$, varies between 50 and 350, and the Weber number, $\mathrm{We}=\rho U^2 a/\gamma$, varies between 7.6×10^{-2} and 2. Here, ρ and η are the liquid density and viscosity, a is the bubble radius, U is the terminal velocity, and γ is the surface tension. The analysis is simplified in this limit because at high Reynolds numbers the fluid flow around the bubble can be analyzed using the potential flow approximation, and at low Weber numbers the deformation of the bubbles can be neglected (see Moore[1] and Levich[2]).

The potential flow interaction between a pair of nondeformable bubbles was analyzed by Kumaran and Koch,[3] who used a perturbation expansion in the parameter (a_i/R) for the velocity potential, where a_i is the radius of bubble i and R is the distance between their centers. They found that the leading-order acceleration due to the potential flow interaction decreases as $(1/R)^4$. A perturbation analysis carried out by Kumaran[4] indicated that the effect of surface deformation on the bubble motion is small for Weber numbers less than 1.0, and the effect of hydrodynamic interactions on the drag force can be neglected for Reynolds numbers larger than 100.

The interaction between a pair of bubbles of different sizes rising due to gravity was also analyzed in Kumaran and Koch.[3] A salient feature of the interaction is that the bubbles repel each other when oriented along the vertical direction, and their surfaces do not come into contact even when the larger bubble is directly below the smaller one. However, the interaction is qualitatively very different when the bubbles are of nearly equal size, i.e., when the ratio of the bubble radii is less than 1.1 at a Reynolds number of 100, and less than 1.07 at a Reynolds number of 200. A numerical calculation of the interaction in Sec. II shows that the bubbles approach each other along the horizontal plane and "collide," i.e., the bubbles have a finite velocity toward each other when their surfaces touch (see Fig. 2). The perturbation analysis does not converge when the distances between the surfaces are small, and the fluid velocity and pressure in the gap between the surfaces become large.

A scaling analysis of the gap flow between two nondeformable bubbles[4] indicated that the dissipation of energy in the gap is $O(\mathrm{Re}^{-1})$ smaller than the kinetic energy of fluid flow. Therefore, a collision could cause the bubble to either coalesce or bounce off each other. A numerical study of the gap flow between deformable bubbles with $\mathrm{We} < 10^{-2}$ by Chesters and Hofman[5] showed that the flow in the gap is not sufficient to arrest the motion of the bubbles before the liquid film becomes unstable and ruptures. At higher Weber numbers, the authors speculated that surface deformation could cause the bubbles to bounce off each other. The bouncing of bubbles at higher Weber numbers was observed by Kirkpatrick and Lockett.[6]

If we assume that the first collision does not lead to coalescence, the bubbles collide repeatedly thereafter. The amplitude of the oscillations decreases due to viscous drag, and the bubbles rise as a horizontally aligned pair after a time period comparable to the viscous relaxation time. A scaling analysis of the gap flow in this configuration by Kumaran[4] indicated that coalescence occurs in finite time, because the force in the small gap between the bubbles is small compared to the force exerted by the outer flow that tends to push the bubbles toward each other. Thus we conclude that even if the first collision does not result in coalescence, the pair of bubbles will coalesce after the energy of their relative motion is dissipated by viscous drag.

The frequency of coalescence in a polydisperse suspension due to interactions between bubbles of nearly equal size is calculated in Sec. III using ensemble averaging.

The authors recently became aware of theoretical and experimental studies of the interaction between a pair of equal sized bubbles by Kok.[7,8] He used an energy conservation formulation to derive equations for the motion of the bubbles. The trajectories calculated using these equations are in agreement with those calculated in Sec. II. Kok's experimental studies showed that the interaction between a pair of equal sized bubbles in pure water leads to coalescence,[7] a result which is in agreement with the theoretical results of Sec. II. Kok also calculated the collision cross section for interactions between a pair of equal sized bubbles in the absence of gravitational and drag forces,[8] using a procedure similar to that used in Sec. III.

II. INTERACTION BETWEEN A PAIR OF BUBBLES OF EQUAL RADII RISING DUE TO GRAVITY

The acceleration of a bubble due to potential flow interaction with another bubble was calculated in Kumaran and Koch.[3] The equations for the acceleration of a bubble i, having radius a_i, are

$$\frac{dU_{iz}}{dt}=\frac{9a_i^3s^3}{2R^4}(\cos\theta\, f+\sin\theta\, g)-\frac{(U_{iz}-U_{it})}{\tau_{vi}}, \qquad (1a)$$

$$\frac{dU_{ix}}{dt}=\frac{9a_i^3s^3}{2R^4}(\sin\theta\, f-\cos\theta\, g)-\frac{U_{ix}}{\tau_{vi}}, \qquad (1b)$$

where

$$f=(\sin\theta\, U_{iz}-\cos\theta\, U_{ix})(\sin\theta\, U_{kz}-\cos\theta\, U_{kx})-2(\cos\theta\, U_{kz}+\sin\theta\, U_{kx})^2,$$

$$g=(\cos\theta\, U_{kz}+\sin\theta\, U_{kx})(\sin\theta\, U_{kz}-\cos\theta\, U_{kx}+\sin\theta\, U_{iz}-\sin\theta\, U_{ix}),$$

$$k=3-i.$$

Here R is the separation of the centers of the bubbles, θ is the azimuthal angle between the vertical and the line joining the centers of the bubbles, s is the size ratio (a_{3-i}/a_i), U_{it} is the terminal velocity of a bubble of species i, and the viscous relaxation time τ_{vi} is the ratio of the drag coefficient and the effective mass of the bubble. The effective mass of the bubble is half the mass of the fluid displaced by it, and the drag coefficient is given by $(12\pi\eta a_i)$ in potential flow. Thus the viscous relaxation time τ_{vi} is

$$\tau_{vi}=\frac{\rho a_i^2}{18\eta}=\frac{\mathrm{Re}_i}{18}\frac{a_i}{U_{it}}. \qquad (2)$$

Here, ρ and η are the liquid density and viscosity, and Re_i is the Reynolds number based on the bubble radius and its terminal velocity.

Figure 1 shows the coordinate system used to analyze the interaction between a pair of equal sized bubbles which have radii a and terminal velocities U_t. In the initial configuration, the bubbles are rising at their terminal velocities, and the subsequent trajectories are generated by numerically integrating the equations for the bubble acceleration (1).

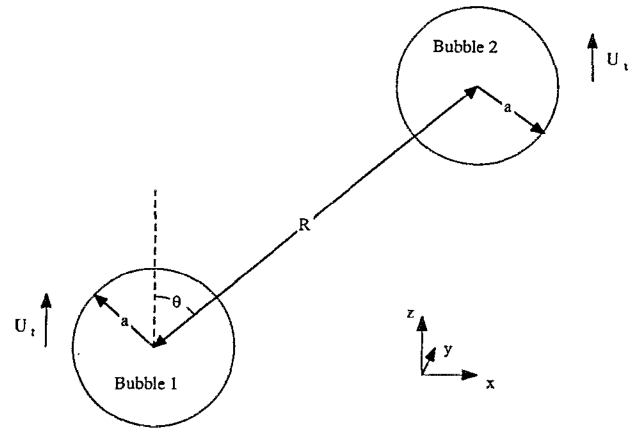

FIG. 1. Coordinate system for analyzing the trajectories of bubbles of equal size that were initially rising at their terminal velocities.

The trajectories of a pair of identical bubbles, rising at a Reynolds number of 200 for different initial configurations are shown in Figs. 2 and 3. The initial distance between them is (5a), and the initial orientation of the line joining the centers is 90° to the vertical in Fig. 2, 60° to the vertical in Fig. 3 (solid line), and 30° to the vertical in Fig. 3 (broken line). The qualitative features of the trajectories are the same in all cases. The bubbles approach each other along the horizontal direction and collide. In Figs. 2 and 3, the trajectories were continued by assuming an elastic bounce after each collision. This will be discussed in detail later.

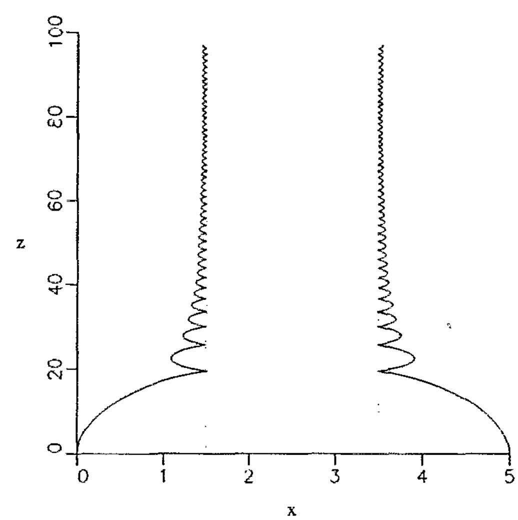

FIG. 2. Trajectories of equal sized bubbles rising due to gravity. The Reynolds number based on bubble radius and terminal velocity is 200. In the initial configuration, the bubbles were horizontally aligned and rising at their terminal velocities, and the distance between their centers was five bubble radii.

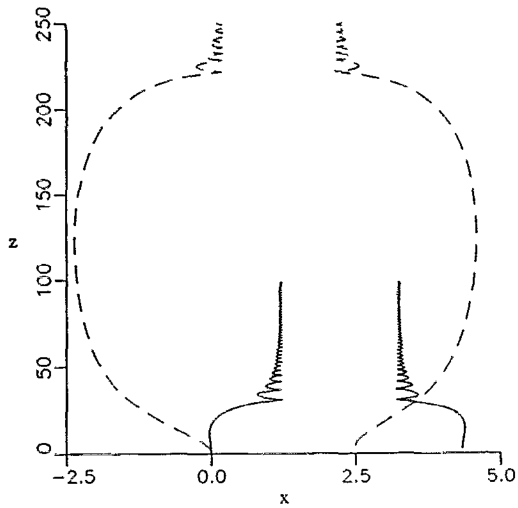

FIG. 3. Trajectories of equal sized bubbles rising due to gravity. The Reynolds number based on bubble radius and terminal velocity is 200. In the initial configuration the bubbles were rising at their terminal velocity and the distance between their centers was five bubble radii. The initial azimuthal angle of the line joining their centers is 60° to the vertical for the solid trajectory, and 30° to the vertical in the broken trajectory.

As the initial orientation of the line joining the bubbles comes closer to the vertical axis, the bubbles take far longer to approach each other. This can be qualitatively explained as follows. Assuming the two bubbles are rising at their terminal velocities, the direction of the acceleration of a bubble at a position **x** due to the presence of a bubble at the origin is shown in Fig. 4. The bubbles repel each other along the vertical direction and attract along the horizontal plane, and they also experience a rotational motion about each other which tends to align them horizontally.

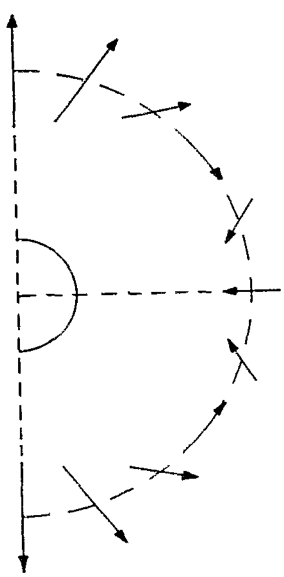

FIG. 4. Orientation of the acceleration of a bubble, whose center is located on the broken line, due to the presence of a bubble at the origin. The bubbles are of equal size, and both bubbles are rising at their terminal velocities.

When the distance between the surfaces of the bubbles is small compared to their radius, a scaling analysis of the gap flow[4] indicates that the viscous energy dissipation during a collision is $O(\mathrm{Re}^{-1})$ smaller than the kinetic energy of fluid flow under high Reynolds number conditions. An earlier study of the gap flow between deformable bubbles by Chesters and Hofman[5] suggested that the bubbles could bounce off each other during a collision if the Weber number is greater than 0.5. At lower Weber numbers, a numerical analysis of the gap flow by Chesters and Hofman[5] showed that the bubbles coalesce. In the analysis, we continue the trajectories by assuming that the bubbles bounce. Because the energy dissipation in the gap is small, the bounce would conserve energy. In the horizontal orientation, conservation of energy gives an exact reversal of velocities as in a bounce of elastic particles. In trajectories calculated for different initial conditions, the maximum deviation of the line joining the centers from the horizontal plane at the point of collision was about 2°, and the error due to the elastic collision assumption is small.

After the first bounce, the bubbles collide repeatedly due to the attraction between horizontally aligned bubbles, as shown in Figs. 2 and 3. The amplitude of the oscillation decreases due to the viscous drag over time scales comparable to the viscous relaxation time τ_v, and the bubbles rise as a horizontally aligned pair. A scaling analysis of the gap flow in this configuration[4] indicted that coalescence occurs within a time period of $O(a/U_t)$, which is $O(18/\mathrm{Re})$ smaller than the viscous relaxation time. Thus, coalescence may be expected to occur eventually even if the speed of the initial collision is large enough to cause a bounce.

The qualitative nature of the interaction between equal sized bubbles is significantly different from the nature of the interaction between unequal sized bubbles, which was analyzed in Kumaran and Koch.[3] In the former, the bubbles approach each other along the horizontal plane and collide repeatedly, whereas in the latter the bubbles repel and their surfaces never come into contact during an interaction. This difference can be qualitatively explained as follows. Potential flow interactions cause repulsion between vertically oriented bubbles and attraction between horizontally oriented bubbles (see Fig. 4). Bubbles of equal size, which have equal terminal velocities, eventually come together along the horizontal plane and collide with a finite velocity toward each other (see Figs. 2 and 3).

When the difference in the radii of the bubbles is smaller than a critical value, we observe the qualitative features discussed above when the horizontal distance between their centers is lower than a critical distance prior to the interaction. The interaction between bubbles whose size difference exceeds the critical value is driven by the difference in their terminal velocities, and takes place within the finite time period required for the bubbles to translate past each other. The interaction is strongest when the horizontal distance between the bubbles is small, and in this configuration the interaction is repulsive. Due to this repulsion the bubbles are pushed apart as the larger bubble translates past the smaller one. The bubbles do experience some attraction farther downstream along the trajectory

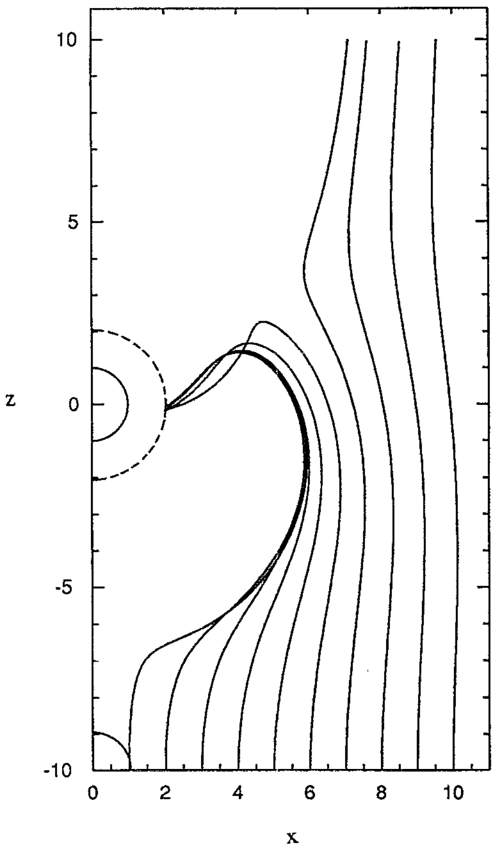

FIG. 5. Interaction between bubbles of nearly equal size. The size ratio s is 1.05, the Reynolds number based on the smaller bubble is 200, and the trajectory of the larger bubble is tracked in a reference frame moving with the smaller one. The broken line shows a surface of radius $(s+1)$; the two bubbles are in contact when the center of the larger bubble touches this surface. The critical impact parameter for this case, calculated from (8b), is 5.9.

when the line joining their centers is oriented closer to the horizontal. However, this attraction is insufficient to capture them in closed trajectories before they have translated past each other.

The results of this section are in agreement with the results of a study of Kok.[7] That author used an energy conservation equation to derive the equations of motion for a pair of equal sized bubbles rising due to gravity. The numerical calculations of the bubble trajectories indicated that the bubbles approach each other along the horizontal plane. The author's experimental studies showed that the interaction between a pair of equal sized bubbles of 1 mm diameter in pure water does lead to coalescence.

Our numerical calculations indicate that the interaction between bubbles of nearly equal size (size ratio less than about 1.07 at Reynolds number of 200, and less than 1.1 at a Reynolds number of 100) also involve repeated collisions if the horizontal distance between the centers of the bubbles is smaller than a critical value when the larger bubble approaches the smaller one. Figure 5 shows an interaction in which the radius of the larger bubble is 1.05 times that of the smaller one, and the Reynolds number based on the radius and terminal velocity of the smaller bubble is 200. The bubbles collide if the horizontal distance between their centers is less than about 6 radii when their separation is large. Empirical relations for this critical distance are derived later. The evolution of the size distribution in a slightly polydisperse suspension due to interactions between bubbles of nearly equal size is analyzed in Sec. III. Interactions in a bidisperse suspension, where the size ratio is larger, do not lead to collisions. However, these interactions cause velocity fluctuations and spatial displacements of bubbles in the suspension. The moments of the velocity distribution in a uniform suspension and the hydrodynamic diffusivities in a nonuniform suspension due to these interactions were analyzed in Kumaran and Koch.[3]

III. EFFECT OF INTERACTIONS ON SIZE DISTRIBUTION

In this section, we estimate the frequency of coalescence between bubbles of nearly equal size in a suspension due to pair interactions, assuming that every collision leads to coalescence. This assumption is valid if the viscous relaxation time, τ_{vi}, which is the maximum time required for the coalescence between a pair of bubbles, is small compared to the time between successive interactions. In a polydisperse suspension, the frequency of interactions between a pair of bubbles scales as (VU_t/a), where V is the volume fraction, and the viscous relaxation time is given by (2). Therefore, the viscous relaxation time is small compared to the time between successive interactions in the limit $V \ll 18/\mathrm{Re}$, and results obtained in this section are valid in this limit. At higher volume fractions, the interaction between a pair of bubbles may be disturbed by the approach of a third bubble before coalescence. Coalescence in dilute suspensions of drops in the atmosphere has been widely studied and is typically expressed in terms of an evolution equation for the size distribution.[9]

The size distribution function $n(a)$ is defined as follows: $n(a)\ da$ is the number of bubbles having radii in the interval da about a per unit volume. Note that $n(a)$ has units of (number/volume/radius), and is not a number density of bubbles. The conservation equation for the size distribution is

$$\frac{dn(a)}{dt} = \int_{a_1} da_1\, n(a_2)\nu(a_2,a_1) - \int_{a_3} da_3\, n(a)\nu(a,a_3). \tag{3}$$

Here $a_2^3 = a^3 - a_1^3$, and $\nu(a,a_1)da_1$ is defined as the frequency of collisions of a bubble of radius a with other bubbles having radii in the interval da_1 about a_1. Note that $\nu(a,a_1)$ has units of collision frequency per unit radius.

The collision frequency is estimated as follows. Consider an interaction between two bubbles having radii a and a_1, respectively. The occurrence of a collision between the bubbles depends on the impact parameter b, which is the horizontal distance between the bubbles before the advent of the interaction (see Fig. 1). If the impact parameter is large, the acceleration due to the interaction is insufficient

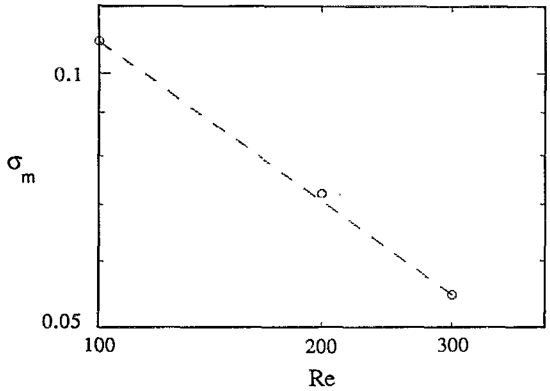

FIG. 6. The maximum size difference at which collisions were observed, σ_m, as a function of the Reynolds number, Re. The dotted line shows the power-law relation (5) for $c_3=1.73$.

to cause a collision in the time period required for the bubbles to translate past each other. If the impact parameter is smaller than a critical impact parameter b_c, however, the interaction causes repeated collisions and coalescence. The frequency of collisional interactions of a bubble of radius a with others of radius in the interval (da_1) about a_1 is given by

$$\nu(a,a_1)da_1=n(a_1)(\pi b_c^2)\Delta U_t\,da_1, \tag{4}$$

where ΔU_t is the difference in the terminal velocities of the bubbles, and the factor (πb_c^2) is commonly referred to as the collision cross section.

Equations (3) and (4) may be simplified by exploiting the fact that the difference in the radii of the bubbles that coalesce is small. The dimensionless size difference is defined as $\sigma=(|a_1-a|/a)$. The maximum size difference, σ_m, at which collisions were observed in the trajectory calculations is plotted in Fig. 6. The results are well represented by the power-law relationship:

$$\sigma_m=c_3\,\mathrm{Re}^{-3/5}, \tag{5}$$

where the constant c_3 for the best fit is 1.73. The difference in the sizes of two coalescing bubbles is asymptotically small for large Reynolds numbers, so that the source term in (3) comes primarily from collisions of two bubbles with nearly half the volume of a bubble of radius a and the sink comes from collisions of two bubbles with radii close to a. Thus, the number densities in (3) can be expanded in a Taylor series about $a_1=(a/2^{1/3})$, $a_2=(a/2^{1/3})$, and $a_3=a$, giving to leading order:

$$\frac{dn(a)}{dt}=n\left(\frac{a}{2^{1/3}}\right)\nu\left(\frac{a}{2^{1/3}}\right)-2n(a)\nu(a), \tag{6}$$

where

$$\nu(a)=n(a)\int_{-\sigma_m}^{\sigma_m}d\sigma(\pi b_c^2)\,|\Delta U_t|, \tag{7}$$

and ΔU_t is the difference in the terminal velocity between a bubble of radius a and one of radius $a(1+\sigma)$.

In (7), there are two undetermined parameters—the critical impact parameter b_c, and the domain of integration for the variable σ. These are estimated using scaling arguments in the following analysis. From the scaling of the terms in the equations for the acceleration (1), we derive power-law relations that express the coalescence frequency as a function of the difference in the radii and the Reynolds number. The coefficients in the relations are calculated from numerical calculations of the trajectories of the bubbles during an interaction.

The critical impact parameter b_c is a function of the size difference between the bubbles, σ, and the Reynolds number Re. This is determined from the scaling of the terms in the acceleration equations (1). For an interaction with impact parameter b, the time required for the larger bubble to translate past the smaller one is $O(b/\Delta U_t)$. If this time period is large compared to the viscous relaxation time, τ_v, the horizontal velocity U_x is $O[\tau_v(U_t^2/a)(a/b)^4]$ from (1b). The interaction could result in a collision if the displacement of the bubble during the interaction, which scales as $(U_x b/\Delta U_t)$, is of the same order of magnitude as the impact parameter b. Thus, the critical impact parameter b_c is $O\{a[\mathrm{Re}/(18\sigma)]^{1/4}\}$ for an interaction whose time period is long compared to the viscous relaxation time.

If the period of interaction, $(b/\Delta U_t)$, is small compared to the viscous relaxation time, the fluctuating velocity U_x during the interaction scales as $[(b/\Delta U_t)\times(U_t^2/a)(a/b)^4]$. The interaction could lead to a collision if the displacement of the bubble, which scales as $(U_x b/\Delta U_t)$, is comparable to b. This indicates that the critical parameter b_c is proportional to $\sigma^{-2/3}$ for interactions whose time period is small compared to τ_v. Thus there are two power-law regimes depending on the relative magnitudes of the viscous relaxation time and the time period of the interaction:

$$b_c=c_1(\mathrm{Re}/\sigma)^{1/4}\quad\text{for}\quad(b_c/\Delta U_t)>\tau_v, \tag{8a}$$

$$b_c=c_2\sigma^{-2/3}\quad\text{for}\quad(b_c/\Delta U_t)<\tau_v. \tag{8b}$$

Here, the constants c_1 and c_2 are to be determined from the trajectory calculations.

The critical impact parameter was determined numerically for various values of the size difference σ at Re=100, 200, and 300. In the numerical procedure, the initial velocities of the bubbles were set equal to their terminal velocities when the center of the larger bubble was a large distance below the smaller one. The trajectories of the bubbles were calculated by integrating the acceleration equations (1) for various impact parameters. The critical impact parameter b_c was determined as the maximum impact parameter for which collisions occurred, and was calculated correct to $\pm 0.05a$. The critical impact parameter is plotted as a function of the size difference σ in Fig. 7. The data show the expected transition from a slope of $-1/4$ for $(b_c/\Delta U_t>\tau_v)$ to a slope of $-2/3$ for $(b_c/\Delta U_t<\tau_v)$. The dotted lines were obtained using $c_1=0.85$ in (8a) and $c_2=0.8$ in (8b), and these fit the data quite well.

We can now proceed to calculate the frequency of collisions, $\nu(a)$, using (7). The difference in the terminal ve-

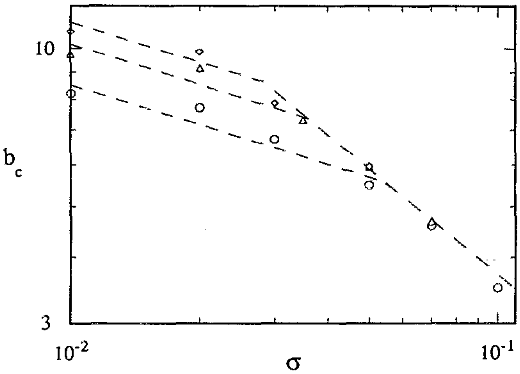

FIG. 7. The critical impact parameter b_c as a function of size difference σ for different Reynolds numbers: ○, Re=100; △, Re=200; ◇, Re=300. The dotted lines show the power-law relations [(8a) and (8b)] for $c_1=0.85$ and $c_2=0.8$.

locities is expanded in a Taylor series in σ, and only the first nonzero term, $U_t=2\sigma U_t$, is retained. The following expression is obtained by substituting $c_1=0.85$, $c_2=0.8$, and $c_3=1.73$ in (8) and (5), and carrying out the integration in (7):

$$\nu(a)=11.207\,\mathrm{Re}^{-2/5}\,n(a)a^3U_t. \quad (9)$$

Both the inertia dominated collisions, (8a), and the viscous relaxation dominated collisions, (8b), produce $O(\mathrm{Re}^{-2/5})$ contributions to the collision frequency in (9). Errors proportional to $\mathrm{Re}^{-1}\,na^3U_t$ and $\mathrm{Re}^{-8/5}\,a^5\,d^2n/da^2$ are produced by truncating the Taylor series expansions for the difference in terminal velocities, ΔU_t, and the number density, $n[a(1+\sigma)]$, respectively.

This completes the derivation of the evolution equation for the number density distribution. As indicated at the beginning of this section, the above analysis is valid in the limit $V \ll 18/\mathrm{Re}$. At higher volume fractions, the interaction between a pair of bubbles is likely to be disturbed by a third bubble before coalescence takes place.

IV. CONCLUSIONS

The effect of interactions on the motion of a pair of equal sized bubbles rising due to gravity was analyzed using the acceleration equations derived in Kumaran and Koch.[3] The interaction causes the bubbles to approach each other along the horizontal plane. If we assume the initial collision does not lead to coalescence, the bubbles collide repeatedly thereafter. The oscillations are damped due to the viscous forces, and the bubbles eventually rise as a horizontally aligned pair whose separation is small compared to their radius. Based on the scaling analysis of the pressure and velocity in the gap between the bubbles by Kumaran,[4] we expect coalescence to occur within a time period that is small compared to the viscous relaxation time.

The repeated collisions and coalescence were also observed when the ratio of the bubble radii is less than 1.1 at a Reynolds number of 100, and less than 1.07 at a Reynolds number of 200, when the impact parameter b is less than a critical value b_c. Here the impact parameter is the horizontal distance between the centers of the bubbles when their separation is large. The frequency of coalescence in a polydisperse suspension was calculated in Sec. III, assuming that every interaction with $b<b_c$ leads to coalescence. This assumption is valid for $V \ll 18/\mathrm{Re}$, where two bubbles that approach each other coalesce before the interaction is disturbed by the approach of a third bubble. The dependence of the critical impact parameter on the difference in bubble sizes and the Reynolds number was deduced from the acceleration equations, and the coefficients in the scaling laws were determined from numerical calculations of bubble trajectories. The leading-order coalescence frequency is proportional to $V\,\mathrm{Re}^{-2/5}$.

ACKNOWLEDGMENT

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