Nonequilibrium Stationary States of a Particle in a Gravitational Field Driven by a Vibrating Surface

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Stationary velocity distribution functions are determined for a particle in a gravitational field driven by a vibrating surface in the limit of small dissipation. It is found that the form of the distribution function is sensitive to the mechanism of energy dissipation, inelastic collisions or viscous drag, and also to the form of the amplitude function of the vibrating surface. The velocity distributions obtained analytically are found to be in excellent agreement with the results of computer simulations in the limit of low dissipation. [S0031-9007(99)08898-5]

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Recent developments in the physics of granular matter [1] have illustrated that the dissipative nature of the interactions between grains can result in a variety of different phenomena. Of particular interest in recent years has been the dynamics of vibrated granular materials [2,3], which exhibit stationary states as well as waves and complex patterns. In order to describe these diverse states of the material, it is necessary to derive macroscopic descriptions by averaging over the microscopic details of the motion of and interactions between individual grains. This goal has proved elusive, however, because a vibrated granular material is a driven dissipative system, and the interactions between the particles are characterized by a loss of energy due to inelastic collisions. The statistical mechanics framework developed for equilibrium or near equilibrium systems cannot be used in this case. Consequently, phenomenological models [4] have been used to describe the dynamics of granular materials. The kinetic theories developed for granular flows [5,6] usually assume that the system is close to "equilibrium" and the distribution function is close to the Maxwell-Boltzmann distribution.

In the search for an understanding of the effects of dissipation on the stationary states, it is useful to study the simplest model that contains many of the features relevant to granular flows, which is a particle in a gravitational field driven by a vertically vibrating surface. One could consider dissipation due to inelastic collisions with the surface and due to fluid drag in this simple example. Even for this model, it has not been possible to derive stationary states by averaging over the interaction between the particle and the surface. The stationary states of a one-dimensional column of particles colliding with a vibrating surface have been studied extensively [7]. In contrast to the single particle case, the dissipation of energy is due to the inelastic binary collisions among the particles in the column. Previous studies of a single particle driven by a vibrating surface usually consider a "thermalizing" base, where it is assumed that the velocity distribution of a particle leaving the base is known [8]. There have also been experiments and simulations of a particle driven by a vibrating surface by Warr *et al.* [9]. The simulations indicated that the distribution function of particle velocities is a Gaussian distribution and the mean-square velocity scales as $T \propto U^2$, where U is the magnitude of the velocity of the vibrating surface. The experiments provided a very different scaling law of the type $T \propto U^{1.04}$. In the present analysis, the stationary distributions are derived for a single particle colliding with a vibrating surface using a realistic microscopic model for the interaction of the particle with the surface.

The stationary states are determined by a balance between the source of energy, due to particle collisions with the vibrating surface, and dissipation, due to inelasticity of the collisions with the surface or due to fluid drag. The limit of weak dissipation, where the dissipation of energy during a collision due to inelasticity or between successive collisions due to viscous drag is small compared to the energy of a particle, is considered. It is shown that the stationary distribution function is sensitive to the dissipation mechanism, and to the form of the amplitude function for the vibrating surface. The mean and mean-square velocities of the vibrating surface depend on the form of the amplitude function used for the vibrating surface. The waveform (A) in Fig. 1 is a symmetric amplitude function with $\langle U \rangle_S = 0$. The waveforms (B) and (C) are asymmetric waveforms; (B) always has a constant value velocity

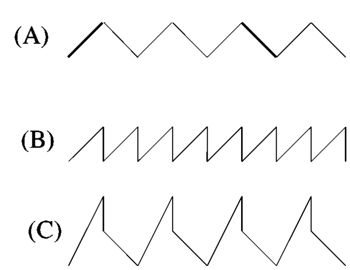

FIG. 1. Waveforms for the amplitude function of the vibrating surface.

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and $\langle U^2\rangle_S = \langle U\rangle_S^2 > 0$, while (C) has $\langle U\rangle_S > 0$ and $\langle U^2\rangle_S > \langle U\rangle_S^2$. Here, $\langle\cdots\rangle_S$ is an average over the distribution of velocities of the vibrating surface. It has been reported earlier [10–12] that the scaling of the mean-square velocity of the particles is a function of the waveform. This analysis indicates that the form of the distribution function is also sensitively dependent on the nature of the waveform.

A two-dimensional coordinate system is used for the analysis, where the y axis is directed opposite to gravity and the x axis is in the horizontal plane. The velocity of the vibrating surface is periodic, and the frequency of oscillations of the surface is large compared to the frequency of particle collisions, so that the velocity of the surface is uncorrelated at successive particle collisions. The validity of these assumptions is examined by comparing the predictions of the analysis with simulation results discussed later. The particle collisions with the surface are nearly elastic $[(1 - e) \ll 1]$, where e is the coefficient of restitution, so that an asymptotic analysis can be used in the small parameter $(1 - e)$. It is useful to describe the stationary state using a distribution $F(u_y)$, such that $F(u_y)du_y$ is the probability that the velocity of a particle that is leaving the vibrating surface is in the interval du_y about u_y. The distribution function $F(u_y)$ is defined only for $u_y > 0$, and the distribution function for the velocity at any height can be inferred from $F(u_y)$, since the particle executes ballistic motion between successive collisions

$$f(y, u_y) = \frac{1}{N} F[(u_y^2 + 2gy)^{1/2}], \tag{1}$$

where N is a suitable normalization factor. The velocity of a particle after a collision with the surface u_y can be related to the velocity before a collision u_y' as follows:

$$u_y - U = -e(-u_y' - U), \tag{2}$$

where U is the velocity of the surface. Note the requirement $U > -u_y'$ for a particle to collide with the surface.

A differential equation for $F(u_y)$ at steady state is determined from the condition that *the "average" accumulation rate in the interval du_y about u_y, due to collisions of the particle with velocity in the interval du_y' about $-u_y'$ with the vibrating surface, is equal to the average depletion rate in the interval du_y about u_y, due to collisions of the particle with velocity in the interval du_y about $-u_y$ with the vibrating surface*. Here, the term "average" denotes an average over the distribution of velocities of the vibrating surface, and u_y' and u_y are related by (2). The resulting conservation equation for the distribution function correct to $O(\epsilon_I)$ and $O(\langle U^2\rangle_S)$ is

$$\epsilon_I\left(2Fu_y + u_y^2\frac{dF}{du_y}\right) - 2\langle U\rangle_S\left(u_y\frac{dF}{du_y} + 2F\right) + 2\langle U^2\rangle_S\left(u_y\frac{d^2F}{du_y^2} + \frac{dF}{du_y}\right) = 0. \tag{3}$$

The stationary distribution function can be determined from the conservation equation (3) as follows: If the average $\langle U\rangle_S = 0$ (symmetric amplitude function), and the conservation equation reduces to

$$\epsilon_I\left(2Fu_y + u_y^2\frac{dF}{du_y}\right) + 2\langle U^2\rangle_S\left(u_y\frac{d^2F}{du_y^2} + \frac{dF}{du_y}\right) = 0, \tag{4}$$

the solution of the above equation is

$$F = \sqrt{\frac{2}{\pi T_{is}}}\exp\left(-\frac{u_y^2}{2T_{is}}\right), \tag{5}$$

where the mean-square velocity T_{is} is

$$T_{is} = (2\langle U^2\rangle_S/\epsilon_I). \tag{6}$$

If $\langle U\rangle_S \neq 0$ (asymmetric amplitude function), it can be seen from (3) that the particle velocity u_y scales as $(\langle U\rangle_S/\epsilon_I)$. In this case, if $\langle U^2\rangle_S \sim \langle U\rangle_S^2$, the leading order equation for the distribution function is

$$\epsilon_I\left(2Fu_y + u_y^2\frac{dF}{du_y}\right) - 2\langle U\rangle_S\left(u_y\frac{dF}{du_y} + 2F\right) = 0. \tag{7}$$

It can easily be verified that the solution for the above equation is a delta function,

$$F(u_y) = \delta\left(u_y - \frac{2\langle U\rangle_S}{\epsilon_I}\right). \tag{8}$$

However, in the vicinity of $u_y = (2\langle U\rangle_S/\epsilon_I)$, the gradient of distribution function becomes large, and the higher order derivatives in the conservation equation could become significant. It turns out that the width of this region is $O(\langle U\rangle_S/\epsilon_I^{1/2})$, and the behavior in this region is determined using the substitution $u_y = (\langle U\rangle_S/\epsilon_I)(2 + \epsilon_I^{1/2}v)$, where v is $O(1)$. The leading order conservation equation for v, which represents the deviation of u_y from $2\langle U\rangle_S/\epsilon_I^{1/2}$, is

$$2\left(\frac{\langle U^2\rangle_S}{\langle U\rangle_S^2} - 1\right)\frac{d^2F}{dv^2} + \epsilon_I^2\left(v\frac{dF}{dv} + F\right) = 0. \tag{9}$$

The solution of the above equation is, once again, a Gaussian distribution,

$$F(v) = \sqrt{\frac{2}{\pi T_{ia}}}\exp\left(-\frac{v^2}{2T_{ia}}\right), \tag{10}$$

where

$$T_{ia} = \frac{2}{\epsilon_I^2}\left(\frac{\langle U^2\rangle_S}{\langle U\rangle_S^2} - 1\right). \tag{11}$$

The above distribution $F(v)$ is valid for $u_y > 0$, and the distribution function for $u_y < 0$ is an image of this about the $u_y = 0$ axis. Therefore, the distribution function at the vibrating surface is a bimodal distribution consisting of two Gaussian distributions centered at $\pm 2\langle U\rangle_S/\epsilon_I$.

The stationary distribution when the dissipation is due to fluid drag is considered next, and the coefficient of restitution for particle-base collisions is set equal to 1. The acceleration of the particle between successive collisions is considered to be of the form

$$\frac{du_y}{dt} = -g - \mu u_y, \tag{12}$$

where the drag force is considered to be linear in the velocity and μ is the ratio of the drag coefficient and the mass of a particle, and has units of inverse time. The particle velocity is large compared to the velocity of the vibrating surface in the limit $(\mu U/g) \ll 1$. Using arguments similar to those used for dissipation due to inelastic collisions, the distribution function can be derived for the present case. The results are as follows: For a vibrating surface with a symmetric amplitude function, the distribution function is obtained by solving the equation

$$u_y^* \frac{d^2F}{du_y^{*2}} + \left(1 + \frac{u_y^{*3}}{3}\right)\frac{dF}{du_y^*} + u_y^{*2}F = 0, \tag{13}$$

where $u_y^* = u_y/(\langle U^2\rangle_S g/\mu)^{1/3}$. For a vibrating surface with an asymmetric amplitude function, the distribution function is

$$F = \sqrt{\frac{2}{\pi T_{va}}}\exp\left(\frac{-w^2}{2T_{va}}\right), \tag{14}$$

where the parameter w is defined as

$$u_y = \left(\frac{\langle U\rangle_S g}{\mu}\right)^{1/2}(\sqrt{3} + \sqrt{\epsilon_D}\, w), \tag{15}$$

$\epsilon_D = (\langle U\rangle_S \mu/g)^{1/2}$, and

$$T_{va} = \frac{\sqrt{3}\,(\langle U^2\rangle_S - \langle U\rangle_S^2)}{2\epsilon_D^2\langle U\rangle_S^2}. \tag{16}$$

The velocity distribution function for the present case is sharply peaked at $\pm(3\langle U\rangle_S g/\mu)^{1/2}$ at the vibrating surface, and the distribution function is $O(1)$ for velocities $O(\epsilon_D\langle U\rangle_S g/\mu)^{1/2}$ different from these peak positions. The distribution function (14) is valid for $u_y > 0$, and the leading order distribution for $u_y < 0$ is a mirror image of this about the $u_y = 0$ axis. Therefore, the distribution function at the vibrating surface is a bimodal distribution consisting of two Gaussian functions centered at $\pm(3\langle U\rangle_S g/\mu)^{1/2}$.

The distribution functions predicted by the analysis were found to be in agreement with computer simulations. In the simulations, the vibrating surface was driven with a waveform of type (A) in Fig. 1 for a symmetric amplitude function, and type (C) for an asymmetric amplitude function. The amplitude of the velocities were $+1$ and -1 for type (A), and $+2$ and -1 for type (C), and the frequency of oscillations was set equal to $20g/\sqrt{T}$, which is 10 times the frequency of collisions for a particle with velocity $\sqrt{T}$. In the simulation, the particle was given an initial velocity, and the velocity of the particle was updated at each collision using the collision rule (2). The velocity was first updated for 10^5 collisions without sampling to remove the dependence on the initial particle velocity at the beginning of the simulation, and samples were than taken for another 4×10^5 collisions in order to determine the distribution function. The results are shown in Figs. 2 and 3 for the different forms of dissipation and for a symmetric amplitude function for the vibrating surface. Similar agreement is found for the asymmetric amplitude form of the amplitude function as well. From the figures, it is seen that there is excellent agreement between the asymptotic result and the simulations when the small parameter in the analysis is 0.01 and 0.03, thereby confirming the validity of the analysis. However, there is some discrepancy between the asymptotic analysis and the numerical simulation for $\epsilon_I \geq 0.1$ for the present case, though it was found that the agreement is improved if the frequency of oscillations of the vibrating surface is increased. In addition, there are discrepancies for a symmetric amplitude function for small values of velocity, because of the invalidity of the assumption that the particle velocity is large compared to the velocity of the vibrating surface.

The dependence of the mean-square velocity on the velocity of the vibrating surface is identical to that obtained [10] for an ensemble of particles driven by a vibrating surface, where the frequency of particle-particle collisions is large compared to particle-wall collisions. However, the form of the distribution function is very different in the two cases, since the leading order distribution function (in the absence of dissipation) is a Maxwell-Boltzmann distribution for all amplitude functions and dissipation mechanisms in that case. The present predictions are also in

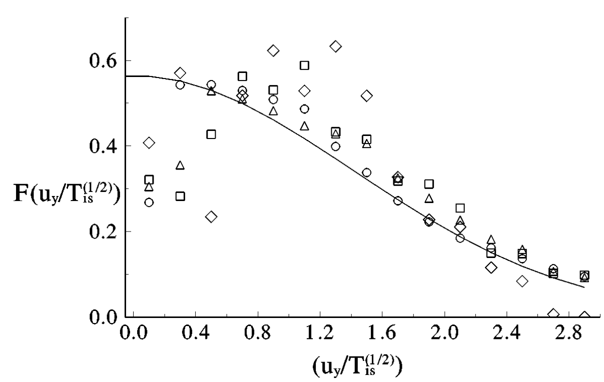

FIG. 2. Velocity distribution $F(u_y/\sqrt{T_{is}})$ as a function of $(u_y/\sqrt{T_{is}})$ for a single particle colliding with a vibrating surface with a symmetric amplitude function, where dissipation is due to inelasticity. Here, T_{is} is given by (6). Solid line: analytical result (5); (○): $\epsilon_I = 0.01$; (△): $\epsilon_I = 0.03$; (□): $\epsilon_I = 0.1$; (◇): $\epsilon_I = 0.3$.

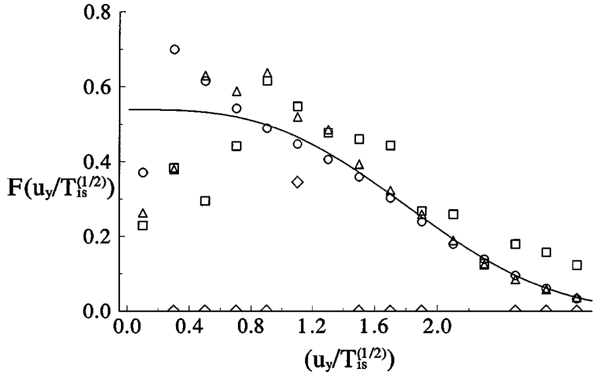

FIG. 3. Velocity distribution $F(u_y^*)$ as a function of $[u_y^* = u_y/(\langle U^2\rangle_S g/\mu)^{1/3}]$ for a single particle colliding with a vibrating surface with a symmetric amplitude function, where dissipation is due to viscous drag. Solid line: solution of Eq. (13); (○): $(\langle U^2\rangle_S \mu^2/g^2)^{1/3} = 0.01$; (△): $(\langle U^2\rangle_S \mu^2/g^2)^{1/3} = 0.03$; (□): $(\langle U^2\rangle_S \mu^2/g^2)^{1/3} = 0.1$; $(\langle U^2\rangle_S \mu^2/g^2)^{1/3} = 0.3$.

agreement with the simulation results of Warr *et al.* [9], where the distribution function for a symmetric waveform and dissipation due to inelasticity was found to be a Gaussian distribution with mean-square velocity proportional to $\langle U^2\rangle_S$. The experimental results of Warr *et al.* [9], however, gave a result of the type $T \propto \langle U^2\rangle_S^{0.52}$. Further analysis [13] indicates that this could result if the drag law is nonlinear, and is of the form $(du_y/dt) = -g - \mu u_y|u_y|$, which is appropriate for turbulent flows.

In summary, stationary velocity distribution functions have been derived for a particle in a gravitational field colliding with a vibrating surface in the limit of small dissipation. It is found that the distribution function is sensitive to the type of dissipation (inelastic collisions or viscous drag), and to the amplitude function of the vibrating surface. These distribution functions have been verified by computer simulations. These distribution functions are rare examples of analytically determined dynamical stationary states in systems far from equilibrium.

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