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Chapter 18

Effect of Collisional Interactions on the Properties of Particle Suspensions

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Velocity distribution functions are determined for a bidisperse sedimenting suspension of particles in a gas and for a sheared suspension of inelastic particles. The distribution functions are determined in two limits. In the kinetic limit, the dissipation of energy due to inelasticity during a collision or viscous drag between successive collisions is small compared to the energy of the particles. In this limit, the distribution function is close to a Maxwell-Boltzmann distribution and the velocity moments are determined using a perturbation expansion about this distribution. In the dissipative limit, the energy dissipation due to inelasticity during a collision or viscous drag between successive collisions is of the same magnitude as the energy of particle velocity fluctuations. In this limit, the distribution function is very different from the Maxwell-Boltzmann distribution and the analytical technique used is specific to the system under consideration.

Introduction

Suspensions of particles in a gas are encountered in many applications, such as solids handling and transport, fluidised beds in chemical unit operations, and in natural systems such as rock slides and snow avalanches. The flow of these suspensions can be broadly classified into two types:

1. In *slow flows*, the distance between the particles is typically small compared to the particle size. In these flows, there is extended contact between the particles, and momentum and energy transport occur due to tangential and normal frictional forces. The examples include flows in bunkers and hoppers in solids handling systems.
2. In *rapid flows*, the particles are widely spaced and the inter-particle distance is usually larger than the particle size. The particles are in vigorous motion, and momentum and energy transfer takes place due to instantaneous particle-particle and particle-wall collisions. These examples include fluidised beds, pneumatic transport and the vigorous motion of a thin layer of particles in rock slides and snow avalanches.

The models currently used for slow flows are continuum models and the constitutive relations are adapted from the yield stress equations used in soil mechanics. At present, no microscopic description is available for slow flows. Microscopic models for rapid flows have been derived by drawing an analogy between the vigorous motion of the particles in a suspension and the fluctuating motion of molecules in a gas which is not at equilibrium. These microscopic descriptions for rapid flows are the subject of the present article.

Microscopic Description of Rapid Flows

The fundamental quantity of interest in a microscopic description of a system of particles is the "distribution function". It provides the density of particles in phase space because mechanical properties such as pressure, shear stress and the energy dissipation rate can be derived from the distribution function. Formally, the single particle distribution function, $f(\mathbf{x}, \mathbf{u}, t)$, is defined such that $nf(\mathbf{x}, \mathbf{u}, t)d\mathbf{x}\, d\mathbf{u}$ is the number of particles whose centres are in the differential volume $d\mathbf{x}$ about the position $\mathbf{x}$, and whose velocities are in the differential volume $d\mathbf{u}$ about $\mathbf{u}$ at time t. Here, $\mathbf{x}$ and $\mathbf{u}$ are the position and velocity, respectively, and n is the number density of the particles. A conservation equation for the single particle distribution function can be written as

$$\frac{\partial(nf)}{\partial t} + \frac{\partial(u_\alpha nf)}{\partial x_\alpha} + \frac{\partial(a_\alpha nf)}{\partial u_\alpha} = \frac{\partial_c(nf)}{\partial t}, \tag{1}$$

where $\mathbf{a}$ is the particle acceleration, Greek subscripts are used to denote the components of a vector, and repeated subscripts represent a dot product. In Eq. (1), the terms on the left-hand side are the time rate of change of the

distribution function, the change in the distribution due to convective transport in real space, and the change due to convective transport in velocity space, respectively. The term on the right-hand gives the change in the distribution function due to particle collisions. This can be written as [1]

$$\frac{\partial_c(nf)}{\partial t} = n^2 \int d\mathbf{k} \int d\mathbf{u}^* [f^{(2)}(\mathbf{x}, \mathbf{u}'; \mathbf{x} + r\mathbf{k}, \mathbf{u}^{*\prime}; t) - f^{(2)}(\mathbf{x}, \mathbf{u}; \mathbf{x} + r\mathbf{k}, \mathbf{u}^*; t)] \times (4\pi r^2 \mathbf{w}.\mathbf{k}), \tag{2}$$

where r is the particle radius, $\mathbf{k}$ is the unit vector in the direction of the line joining the centres of the particles at the point of collision, $\mathbf{u}$ and $\mathbf{u}^*$ are the velocities of the particles before collisions, $\mathbf{u}'$ and $\mathbf{u}^{*\prime}$ are the velocities of the particles after collision, and $\mathbf{w} = \mathbf{u} - \mathbf{u}^*$ is the difference in the velocities of the colliding particles.

Equation (1) for the single particle distribution, $f(\mathbf{x}, \mathbf{u}, t)$, can only be solved if the pair distribution function, $f^{(2)}(\mathbf{x}, \mathbf{u}; \mathbf{x}+r\mathbf{k}, \mathbf{u}^*, t)$, is known. However, a conservation equation for the pair distribution function contains three particle distribution functions. In general, a conservation equation for an n particle distribution function contains an $(n + 1)$ particle distribution function, and one obtains an infinite hierarchy of equations known as the BBKGY hierarchy [1]. General methods for solving this hierarchy of equations are not available. In the kinetic theory of gases, the positions and velocities of the colliding particles are considered to be uncorrelated (the assumption of "molecular chaos") and the two particle distribution functions are the product of the single particle distribution function. This assumption is valid when the mean free path, which is the distance a molecule travels between successive collisions, is large compared to the size of the molecule. Hence there are no repeated collisions between the same pair of particles. With this assumption, a closed equation for the single particle distribution function, called the "Boltzmann equation", is obtained [2]:

$$\frac{\partial(nf)}{\partial t} + \frac{\partial(u_\alpha nf)}{\partial x_\alpha} + \frac{\partial(a_\alpha nf)}{\partial u_\alpha} = n^2 \int d\mathbf{k} \int d\mathbf{u}^* [f(\mathbf{x}, \mathbf{u}')f(\mathbf{x} + r\mathbf{k}, \mathbf{u}^{*\prime}, t) - f(\mathbf{x}, \mathbf{u})f(\mathbf{x} + r\mathbf{k}, \mathbf{u}^*, t)](4\pi r^2 \mathbf{w}.\mathbf{k}). \tag{3}$$

This equation is a non-linear integro-differential equation, and is difficult to solve in general. However, it can be shown [2] that for a system at steady state, in the absence of external forces, the distribution function is a

Maxwell-Boltzmann (MB) distribution. For dense systems where the particle positions are correlated before a collision, such a simplification is not possible and some sophisticated mathematical techniques, called cluster expansions, have been developed for a dense system with hard-sphere molecules. There are no equally successful methods for a dense system of particles. In the present study, we will deal exclusively with dilute suspensions and the Boltzmann equation is the starting point of the description.

While drawing an analogy between gases and dilute suspensions, it should be noted that there is an important difference: the energy of the molecules in a gas at equilibrium is conserved but the motion of the particles in a suspension can only be sustained if there is a continuous source of energy. This is because dissipation exists due to inelastic collisions or the drag force of the gas. Based on this distinction, there are two limiting cases for the dynamics of a dilute suspension:

1. The dynamics will resemble that of the molecules of a gas if the dissipation of energy during a binary collision (due to inelasticity), or between successive collisions (due to viscous drag), is small compared to the average energy of the particle velocity fluctuations. This limit is known as the *kinetic* limit.
2. In the complementary limit, called the *dissipative* limit, the change in energy during a collision or between successive collisions is of the same magnitude as the energy of fluctuations. The properties are very different from that of a gas at equilibrium.

The dynamics of suspensions in the kinetic limit are obtained by assuming that the distribution function is a small perturbation about the MB distribution for a gas at equilibrium. The deviations from the MB distribution are evaluated from the velocity moments of the Boltzmann equation. These methods are fairly standard and are therefore not discussed in detail here. However, the extension of the Boltzmann H-Theorem to dissipative systems and its consequences and the calculation of the distribution function for suspensions in the dissipative limit are examined in the next section. Two systems, a bidisperse suspension of particles settling in a gas and a sheared suspension of inelastic particles, are considered. The analysis is restricted to spatially homogeneous suspensions at steady state. Hence the distribution function is independent of time and the spatial co-ordinates.

Distribution Function in the Kinetic Limit

In the kinetic limit, the collisional transport in phase space (the two terms on the right-hand side of Eq. (3)), which represents the rate of transport of particles into and out of a differential volume in velocity space, is large compared to the terms on the left-hand side. In addition, the particles are nearly elastic. In this case, a perturbation expansion can be used where the system is considered to be a collection of elastic particles in the leading approximation, and the effects of inelasticity, drag and body forces are included in higher order corrections in a systematic fashion. The leading order distribution function is a Maxwell-Boltzmann distribution [2]:

$$F = \frac{1}{(2\pi T)^{3/2}} \exp\left(\frac{-mc_\alpha^2}{2T}\right) \tag{4}$$

where m is the mass of the particle, the fluctuating velocity $c_\alpha = u_\alpha - U_{m\alpha}$ is the difference between the particle velocity and the mean velocity, and T is the "temperature". Unlike the case of molecular gases, the temperature is not specified *a priori*, but is determined by a balance between the source and dissipation of energy.

Bidisperse Particle-Gas Suspension

The system consists of a suspension of particles with masses m_1 and m_2, radii r_1 and r_2, and terminal velocities $\mathbf{U}_1$ and $\mathbf{U}_2$ settling in a gas. The drag force on the particles is considered to be a linear function of the particle velocity and the acceleration is

$$a_{i\alpha} = -(\mu_i/m_i)(u_{i\alpha} - U_{i\alpha}), \tag{5}$$

where the drag coefficient, μ_i, is $(6\pi\eta r_i)$ in the Stokes regime. The inertia of the gas is neglected compared to that of the particle. Hydrodynamic interactions are also neglected so that the dominant effects comprise the inertia of the particles and viscous drag due to the gas. There are two important time scales: the viscous relaxation time, $\tau_v = (m_1/\mu_1)$, which is the time taken by a particle to relax to its terminal velocity after a collision, and the collision time, $\tau_c = (1/(n_1 d_{12}^2 c))$, is the time that elapsed between successive collisions. Here, $d_{ij} = r_i + r_j$ and c is the magnitude of the fluctuating velocity. In the kinetic limit, the collision time is small compared to the viscous relaxation time.

As noted earlier, the distribution function is a Maxwell-Boltzmann distribution in the leading approximation. The mean velocities and "temperatures" for the two species are also equal. The first correction to the distribution function can be obtained using an asymptotic analysis where the distribution function is expressed as

$$f_i(\mathbf{c}_i) = F_i(\mathbf{c}_i)[1 + \delta\Phi_i(\mathbf{c}_i)]\,. \tag{6}$$

The small parameter, δ, will be specified a little later. When this is inserted into the Boltzmann equation, a linear equation for the perturbation, ϕ_i, is obtained:

$$\begin{aligned}\frac{\partial(a_{i\alpha}F_i)}{\partial c_{i\alpha}} = \delta\sum_{j=1}^{2} n_j \int d\mathbf{k} \int d\mathbf{c}_j : [F(\mathbf{c}_i)F(\mathbf{c}_j)[\Phi(\mathbf{c}'_i) \\ + \Phi(\mathbf{c}'_j) - \Phi(\mathbf{c}_i) - \Phi(\mathbf{c}_j)](\pi d_{ij}^2 \mathbf{w}.\mathbf{k})\,,\end{aligned} \tag{7}$$

where $\mathbf{c}_i$ and $\mathbf{c}_j$ are the velocities of the colliding particles before the collision and $\mathbf{c}'_i$ and $\mathbf{c}'_j$ are the velocities after the collision. Note that the velocity co-ordinate has been transformed from the particle velocity, $\mathbf{u}_i$, to the fluctuating velocity, $\mathbf{c}_i$. This transformation is trivial because the mean velocities of the two species are equal in the leading approximation.

Equation (7) does not provide the magnitude for δ. However, it can be obtained from the equivalent of the Boltzmann H-Theorem [2] for this system. The function, H, is defined as

$$H = \sum_i \int d\mathbf{x} \int d\mathbf{c}_i f_i \log(f_i)\,. \tag{8}$$

The time derivative of H is

$$\begin{aligned}\frac{dH}{dt} &= \sum_i \int d\mathbf{c}_i \left[\int d\mathbf{x}(1+\log(f_i))\frac{\partial f_i}{\partial t}\right] \\ &= \left[\sum_i \int d\mathbf{x} \int d\mathbf{c}_i \left((1+\log(f_i))\frac{\partial_c f_i}{\partial t} - f_i\frac{\partial a_{i\alpha}}{\partial c_\alpha} - \underline{\frac{\partial}{\partial c_\alpha}(\mathbf{a}_i f_i \log f_i)}\right)\right].\end{aligned} \tag{9}$$

The underlined term in the above expression can be reduced to a surface integral in velocity space which is zero, and therefore

$$\frac{dH}{dt} = \sum_i \int d\mathbf{x} \int d\mathbf{u}_i \left((1+\log(f_i))\frac{\partial_c f_i}{\partial t} - f_i\left(\frac{\partial a_{i\alpha}}{\partial c_\alpha}\right)\right). \tag{10}$$

When the viscous relaxation time is large compared to the time that elapsed between collisions, the asymptotic expansion in Eq. (6) can be used for the distribution function. The leading order equation for the rate of change of H is

$$\left.\frac{dH}{dt}\right|_0 = \sum_{ij} \int d\mathbf{x} \int d\mathbf{k} \int d\mathbf{u}_i \int d\mathbf{u}_j \left[(F'_iF'_j - F_iF_j)\log\left(\frac{F_iF_j}{F'_iF'_j}\right)(\mathbf{w}.\mathbf{k})\right]. \tag{11}$$

It can be shown [2] from the above equation that the leading order distribution function is a Maxwell-Boltzmann distribution, Eq. (7), from the Boltzmann H-Theorem. The first correction to $(dH/dt)_1$ is

$$\begin{aligned}\left.\frac{dH}{dt}\right|_1 = -\delta^2 \sum_{ij}\int d\mathbf{x}\left[\int d\mathbf{c}_i \int d\mathbf{c}_j F_i(\mathbf{c}_i)F_j(\mathbf{c}_j)[(\Phi_i(\mathbf{c}'_i) + \Phi_j(\mathbf{c}'_j)\right. \\ \left. - \Phi_i(\mathbf{c}_i) - \Phi_j(\mathbf{c}_j))^2(\mathbf{w}.\mathbf{k})] - \int d\mathbf{c}_i F_i\left(\frac{\partial a_{i\alpha}}{\partial c_{i\alpha}}\right)\right].\end{aligned} \tag{12}$$

At steady state, the first correction to (dH/dt) is also zero. In the above equation, the first term on the right-hand side is proportional to $\delta^2\tau_c^{-1}$ while the second term is proportional to τ_v^{-1}. Thus, it can be inferred that $\delta = (\tau_c/\tau_v)^{1/2}$. In addition, a comparison of Eqs. (7) and (12) shows that

$$\frac{\partial a_{ia}}{\partial c_\alpha} \sim \frac{\delta a_{ia}}{\tau_v}. \tag{13}$$

This provides the estimate, $c \sim \delta U_m$, for the fluctuating velocity and $T \sim \delta^2 m_i U_m$ for the temperature. (Here, it has been assumed that the mean velocity of the suspension and the terminal velocities of the two species are of the same magnitude.)

The first correction to the distribution function is obtained by solving Eq. (7) using the Enskog expansion [2] for the present case. The following functional form is assumed for Φ_i:

$$\Phi_i(\mathbf{c}_i) = A_i(\mathbf{C}_i)\mathbf{C}_i.\mathbf{U}_m \tag{14}$$

where $\mathbf{C}_i = (m_i^{1/2}\mathbf{c}_i/T^{1/2})$ is $O(1)$. It is not possible to obtain an explicit solution for $A(\mathbf{C}_i)$. However, this can be expanded in an appropriate orthogonal function space and the series solution can be obtained. However, the magnitude of the difference in the mean velocities of the two species can be obtained without explicitly solving the equation. It can be easily seen that the difference between the mean velocity of species, i, and the mean velocity of the suspension, $\int d\mathbf{c}_i F_i \delta\Phi_i$, is $O(\delta^2 U_m)$ since $\mathbf{c}_i \sim \delta U_m$.

To determine the exact values, it is necessary to use the moment expansion method [3] where the Boltzmann equation is multiplied by different moments of the velocity distribution to obtain conservation equations for the velocity moments. However, the present analysis provides a clearer insight into the effect of velocity-dependent forces on the dynamics of the system in the kinetic limit.

Sheared Suspension of Inelastic Particles

The shear flow of a suspension of slightly inelastic particles in the kinetic limit has been studied in detail. The analysis is very similar to that for a gas of hard-sphere molecules in shear flow [2]. Hence the details of the analysis are not discussed here. The major differences are that the collisions between particles are inelastic and the temperature of the suspension is determined by a balance between the input of energy and dissipation due to shear flow and inelastic collisions, respectively. The Enskog expansion is used to determine the deviation of the distribution function from the MB distribution in Eq. (4):

$$f = F(1 + \delta\Phi) \tag{15}$$

where Φ, the deviation from the Maxwell-Boltzmann distribution, has the following form in a shear flow:

$$\Phi = A(\mathbf{C})C_\alpha \partial_\alpha T + B(\mathbf{C})C_\alpha C_\beta(\partial_\alpha U_\beta) \tag{16}$$

where $\mathbf{C} = (m\mathbf{c}/T)^{1/2}$, and $\partial_\alpha U_\beta$ and $\partial_\alpha T$ are the gradients in the mean strain rate and temperature, respectively. Using the above expansion, constitutive equations can be derived for the density, momentum and "temperature" of the suspension. The conservation equation for the granular temperature contains a source term and dissipation term due to the shear work and inelastic collisions, respectively. A balance between these gives the granular temperature at steady state. To extend the analysis to the dense limit, attempts have been made to use a pair distribution function which is not just a single particle distribution function, but also includes the effect of excluded volume. The distribution function most commonly used is the Carnahan-Starling approximation for a system of dense gases.

Distribution Functions in the Dissipative Limit

In contrast to the kinetic limit, no standard methods exist to determine the distribution function in the dissipative limit. The method used has to be designed for the system under consideration. This is illustrated in the examples that follow.

Bidisperse Particle-Gas Suspension

In the dissipative limit, the number density of the particles is sufficiently small. Thus, the viscous relaxation time, $\tau_{vi} = (m_i/\mu_i)$, is small compared to the time that elapsed between successive collisions, $\tau_{cij} = 1/(n_i d_{ij}^2(U_1 - U_2))$. In this limit, a perturbation expansion in the small parameter, $\epsilon = (\tau_{v1}/\tau_{c12})$, is used to calculate the distribution function. In the leading approximation, the effect of collisions is neglected and the particles are considered to settle at their terminal velocities. In this case, the distribution functions are delta functions at the terminal velocities of the two species.

The distribution function that includes the effect of collisions between particles settling at their terminal velocities can be determined using a flux balance in velocity space. The balance equation for the distribution function is

$$\frac{\partial a_{i\alpha} f}{\partial u_{i\alpha}} = N_i^{in}(\mathbf{u}_i) - N_i^{out}(\mathbf{u}_i)\,, \tag{17}$$

where N_i^{in} and N_i^{out} are the flux of particles entering and leaning a differential volume due to collisions. In the collisional limit, the number of particles with velocities $O(U_1 - U_2)$ that are different from their terminal velocities is small. Therefore, in the calculation of the leading order estimate of N_i^{in} and N_i^{out}, it is assumed that the colliding particles are moving at their terminal velocities. The collisional fluxes are determined by relating the angle made by the line joining the centres of the particles at the point of collisions to the change in the velocity. The details of the calculation are found in Kumaran and Koch [7]. The collisional fluxes are inserted into Eq. (17) to determine the distribution function

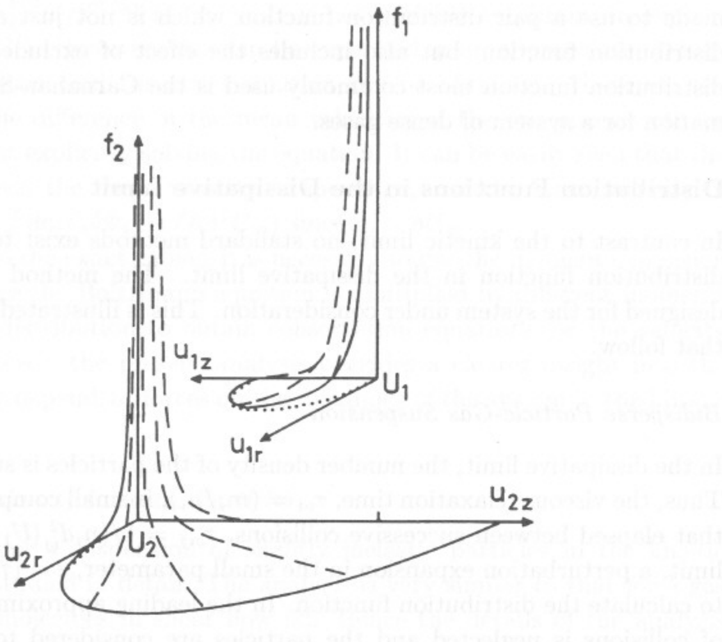

Fig. 1. Schematic of the shape of the distribution function in a bidisperse suspension. The zero levels of the distribution function of the two species have been separated for clarity. The dotted line represents the projection of the surface onto the (u_x, u_y) plane and the solid line shows the distribution function on this surface.

$$f_i = (\epsilon\gamma_{ik}/\pi)(\cos(\chi_i))^{1-\epsilon\gamma_{ik}}(2M_i)^{-\epsilon\gamma_{ik}} v_i^{(\epsilon\gamma_{ik}-3)}, \tag{18}$$

where $\epsilon = (\tau_{v1}/\tau_{c12})$ and $\gamma_{ik} = (\tau_{vi}/\tau_{v1})(\tau_{c12}/\tau_{cik})$.

The distribution function, Eq. (18), shown in Fig. 1, is very different from the MB distribution. Some of its salient features are described here. The distribution is non-zero only in finite regions of the velocity space and has a divergence at the terminal velocities of the two species. The first correction to the velocity moments can be determined using the distribution function, Eq. (18). The analysis shows that the difference between the particle velocity and the mean velocity, $\langle v_z \rangle$, is $O(\epsilon)$ smaller than the terminal velocity in the limit $\epsilon \ll 1$. Meanwhile, the mean square velocities are $O(\epsilon)$ smaller than the square of the terminal velocity. The distribution function is highly anisotropic. The ratio of the mean square velocity in the vertical and horizontal directions four in the limit $\epsilon \to 0$. In addition, the distribution function

is highly skewed. Hence the ratio, $(\langle v_z^3 \rangle / \langle v_z^2 \rangle^{3/2}$, diverges proportional to $\epsilon^{-1/2}$ in the limit $\epsilon \to 0$.

Sheared Suspension of Inelastic Particles

For the sake of simplicity, the system considered here is a two-dimensional suspension of inelastic disks. Nevertheless, the analysis can easily be extended to a three-dimensional suspension of spherical particles. The disks are of radius r, number density n and coefficient of elasticity e in a channel with width L. The channel is bounded by walls at $y = (L/2)$ and $y = -(L/2)$ moving with velocities $+U_w$ and $-U_w$ in the x direction, respectively. Here, the co-ordinate y is perpendicular to the walls of the channel and the x co-ordinate is along the flow direction. The particle-particle collisions are described by the standard laws for collisions between smooth elastic disks. The change in the particle velocity due to a wall collision is given by

$$u'_x - u_x = (1-e_t)(\pm U_w - u_x)_y \; u'_y - u_y = -(1-e_n)u_y\,, \tag{19}$$

where (u_x, u_y) and (u'_x, u'_y) are the particle velocity before and after the wall collision, respectively, and e_t and e_n are the tangential and normal coefficients of restitution which are less than one. In the equation for $u'_x - u_x$, the positive sign for U_w is used for a collision with the wall at $y = +(L/2)$ and the negative sign for the wall at $y = -(L/2)$.

In the dissipative limit, $(nrL) \ll 1$, particle-wall collisions are more frequent than particle-particle collisions. In the absence of inter-particle collisions, a particle with a non-zero velocity in the y direction collides repeatedly with the walls. Its velocity after i collisions evolves as

$$u_x + (-1)^i(1+(-1)^{(i-1)}e_t^i)U = e_t^i u_x^{(0)}, \quad u_y = (-1)^i e_n^i u_y^{(0)}, \tag{20}$$

where $U = (1-e_t)U_w/(1+e_t)$, $u_x^{(0)}$ and $u_y^{(0)}$ are the particle velocities before the first collision, and the first collision is assumed to take place with the wall at $y = +(L/2)$. It can be seen that in the limit of large i, the particle velocity converges towards $(\pm U, 0)$. Consequently, in the absence of particle collisions, it is expected that the velocities of all the particles converge towards $(\pm U, 0)$, which is independent of their initial velocities and depends only on the wall velocity and the coefficients of restitution.

In the limit of small u_y, however, it cannot be assumed that particle-wall collisions are more frequent than particle-particle collisions. The frequency of

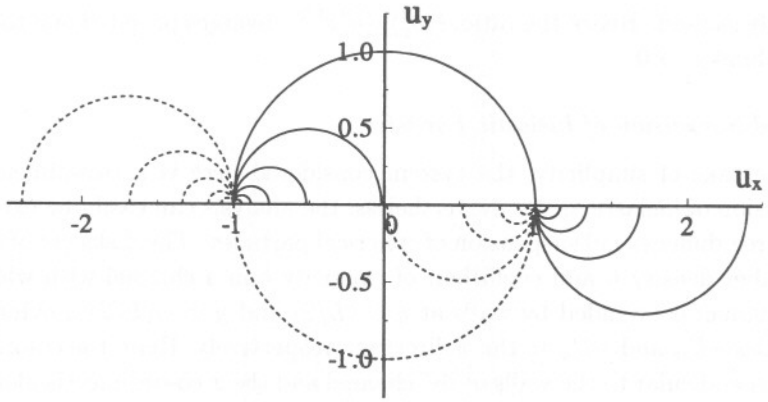

Fig. 2. The contours, C_i, of the particle velocities in the $u_x - u_y$ plane where the index, i, represents the number of times the particle has collided with the walls after it has acquired a velocity in the y direction due to a binary collision. The solid lines show the location of particles whose first collision is with the wall at $y = +(L/2)$ while the broken lines show the location of particles whose first collision is with the wall at $y = -(L/2)$. The coefficients of restitution, e_t and e_n, are both 0.7.

particle-wall collisions per unit length of the channel in the x direction scales as nru_y while that for particle-particle collisions is proportional to n^2r^2LU. This is because the difference in particle velocities scales as U for $u_y \ll U$. Therefore, the frequency of particle-particle collisions is the same as that of particle-wall collisions for $(u_y/U) \sim \epsilon$. To determine the effect of collisions to leading order in small ϵ, it is assumed that half of the particles have velocities $(U, 0)$ and the remaining half velocities $(-U, 0)$ prior to collision. Consider a collision between particle A with velocity $(U, 0)$ and particle B with velocity $(-U, 0)$. The velocity after collision is given by

$$\begin{aligned} u'_{Ax} &= -V\cos(2\theta) \; u'_{Ay} = -V\sin(2\theta)\,, \\ u'_{Bx} &= V\cos(2\theta) \quad u'_{By} = V\sin(2\theta)\,, \end{aligned} \tag{21}$$

where θ is the angle made by the line joining the centres of the particles at the point of collision with the x-axis. Therefore, binary collisions tend to transport particles onto a circle of radius, U, in velocity space as shown in Fig. 2. The subsequent collisions with the walls modify the velocity of the particles as indicated by Eq. (20), so that the velocity after i collisions, $(u_x^{(i)}, u_y^{(i)})$, is given by the parametric relations:

$$\begin{aligned} u_x^{(i)} + (1 + (-1)^{(i-1)}e_t^i)U &= e_t^i U\cos(\chi) \quad \text{for } 0 < \chi < \pi \\ u_x^{(i)} - (1 + (-1)^{(i-1)}e_t^i)U &= e_t^i U\cos(\chi) \quad \text{for } \pi < \chi < 2\pi\,. \\ u_y^{(i)} &= e_n^i U\sin(\chi) \end{aligned} \tag{22}$$

The above equations show that the particle positions are located along the ellipses C_i centred at $(\pm(1 + (-1)^{(i-1)}e_t^i)U, 0)$ with radii $e_t^i U$ and $e_n^i U$ lying along the x and y directions as shown in Fig. 2.

The distribution function along each of these contours is obtained by a flux balance in velocity space. The details of the calculation are not given here. The reader is referred to [8] for the details. The distribution function, $f_i(\chi)$, is defined such that $nf_i(\chi)d\chi$ is the number of particles in the differential angle, $d\chi$, about χ on the contour C_i. This is given by

$$f_i(\chi) = \frac{f_0(\chi)}{e_n^i}\prod_{j=1}^{i}\left[1 + \frac{2\epsilon}{(e_n)^j|\sin(\chi)|}\right]^{-1} \tag{23}$$

where $f_0(\chi)$, the distribution function after the binary collision, is

$$f_0(\chi) = \frac{\epsilon}{2|\sin(\chi)|}\left[\cos\left(\frac{\chi}{2} - \frac{\pi}{2}\right)\right]\,. \tag{24}$$

It can be easily shown that this distribution function is normalised:

$$\sum_{i=0}^{\infty}\int_0^{2\pi} d\chi f_i(\chi) = 1\,. \tag{25}$$

The moments of the velocity distribution function can be easily calculated using the distribution function, Eq. (23). It is found that $\langle u_x^2\rangle \to U^2$ and $\langle u_y^2\rangle \sim V^2\epsilon$ in the limit $\epsilon \to 0$. In addition, the cross-correlation, $\langle u_x u_y\rangle \sim U^2\epsilon\log(\epsilon^{-1})$, and the shear stress decrease proportional to $\epsilon\log(\epsilon^{-1})$ in this limit. The qualitative behaviour of the velocity moments are the same for a three-dimensional suspension of elastic spheres as well as for suspensions of inelastic disks and spheres.

Conclusions

The derivation of the velocity distribution function for dilute particle suspensions in the kinetic and dissipative limits was discussed. In the kinetic limit, the distribution function is close to a Maxwell-Boltzmann distribution for a gas at equilibrium. For a sheared suspension, the asymptotic scheme for determining the distribution function is similar to that used in the Chapman-Enskog theory for dense gases. However, there is a minor difference: the "temperature" is not externally imposed, but is determined by a balance between the source of energy and dissipation due to shearing and inelastic collisions, respectively. For a bidisperse sedimenting suspension, the analysis is different from that used in the Chapman-Enskog theory due to a velocity-dependent drag force. The Boltzmann H-Theorem can be used to show that the magnitude of the fluctuating velocity scales as δU_m, where U_m is the mean velocity of the suspension. The difference in the mean velocity of the two species scales as $\delta^2 U_m$, where the small parameter $\delta \sim (\tau_c/\tau_v)^{1/2}$ with τ_c and τ_v being the time that elapsed between collisions and the viscous relaxation, respectively. Therefore, it can be inferred from the Boltzmann H-Theorem that the fluctuating velocity is small compared to the mean velocity of the suspension.

The distribution function in the dissipative limit is very different from the MB distribution. For a bidisperse sedimenting suspension, the distribution function is non-zero only in a finite region of the velocity space. It also has a divergence at the terminal velocities of the two species. The distribution function is highly anisotropic and the mean square velocity in the vertical direction is four times that in the horizontal direction. In addition, it is highly skewed and the skewness increases proportional to $(\tau_v/\tau_c)^{-1/2}$ in the limit, $\tau_v \ll \tau_c$.

For a sheared suspension of two dimensional disks, the dissipative limit corresponds to the regime $\epsilon \equiv (nrL) \ll 1$, where n is the particle number density, r is the particle radius and L is the width of the channel. In this limit, the frequency of particle-wall collisions is large compared to that of particle-particle collisions. The distribution function is sharply peaked around $(u_x, u_y) = (\pm U, 0)$ and is non-zero only along certain contours in the velocity space. This is shown in Fig. 2. The distribution function is highly anisotropic and the mean square velocity normal to the walls is $O(\epsilon)$ smaller than that in the flow direction. The cross-correlation, $\langle u_x u_y \rangle$, which is proportional to the shear stress, is $O(\epsilon \log(\epsilon^{-1}))$ smaller than the mean square velocity in the flow direction.

The above studies indicate that the distribution function in the kinetic limit is close to a Maxwell-Boltzmann distribution for a hard sphere gas. It can be determined using a perturbation analysis in which the MB distribution is the leading approximation. The distribution function in the dissipative limit is very different from the MB distribution and the analytical technique used is specific to the system under consideration.

Discussion

J. R. A. Pearson To what physical systems and/or phenomena do your approximate theories apply and provide physical insight?

V. Kumaran The theories derived here apply to the rapid flows of suspensions of particles in a gas, such as shear flows or settling suspensions in the kinetic and dissipative limits. In real systems, the flow may be in either of these limits, or in the intermediate regime. If the flow is in either of these limits, the theory applies without modifications. If it is in the intermediate regime, approximate distribution functions, such as the one devised by Kumaran, Tsao and Koch [9] can be used. The present analysis is useful for devising these approximation distribution functions since it provides the limiting behaviour to which any valid solution should converge.

In real systems, different points in the flow have different parameter values. In these cases, it would be necessary to use different distribution functions at different points in the flow to get a complete description. In this sense, the present description has an advantage over continuum descriptions. This is because in the latter, the same description is used throughout the flow even though the flow conditions could be very different.

M. J. Adams Could your method be adapted to describe the behaviour of fluidised beds which show complex behaviour such as the formation of bubbles?

V. Kumaran This analysis cannot be easily adapted to gas fluidised beds due to the complexity of the interaction between the gas and the fluid. The simple Stokes law for the interaction between the particles and the gas would not suffice. A more complete description of the gas-particle interaction at high Reynolds number in dense suspensions would be necessary. In addition, the assumption of molecular chaos would not be a good one for a fluidised bed where the particle density is quite high. However, the present analysis could be used to describe a vibrated fluidised bed where the fluidisation is due to the vibration of the bottom surface of the bed. In this case, the dynamics of the particles can be described by simple laws. It has also been experimentally observed that the density of the suspension is low enough to justify the use of the present theories for dilute suspensions.

M. E. Cates The limit $\epsilon \ll 1$ corresponds to the usual definition of Knudsen flow in gases for which the viscosity is independent of density. Your results are different. Why?

V. Kumaran The difference is in the boundary conditions used for the shear flow. For a gas sheared between two surfaces, the size of the molecules is small compared to that of the surface roughness on the surfaces. Therefore, the stochastic Maxwell boundary condition is used. This is because it is assumed that a fraction of the molecules incident on a surface are reflected elastically while the rest are reflected with a random velocity chosen so that the average temperature of the reflected molecules is equal to the temperature of the surface. In the present system, the size of the particles is large compared to the size of the surface roughness. Hence deterministic boundary conditions are used. This leads to a difference in the behaviour of the two systems.

J. Goddard Tsao and Koch [10] had recently performed an analysis of the shear flow of a particle suspension in which they reported the existence of two states of suspension, that is, an "ignited" and "collapsed" state. How are these states related to the asymptotic limits you talked about?

V. Kumaran The analysis of Tsao and Koch was for the shear flow of a suspension of particles in a gas where it is subjected to a shear flow. The "collapsed" state corresponds to a dilute suspension where most of the particles travel along the streamlines. Collisions also occur due to the relative velocity between particles travelling on different streamlines separated by a distance less than the particle diameter. The analysis of the collapsed state resembles closely the analysis for the dissipative limit of a bidisperse particle suspension discussed here, although the mechanism that induces particle collisions is different. The analysis of the ignited state is very similar to that for the kinetic limit of a bidisperse suspension. Therefore, the "collapsed" and "ignited" states correspond to the dissipative and kinetic limits.

M. Lal Can simulation methods, such as the lattice Boltzmann simulation, be used for these suspensions?

V. Kumaran If one is interested in simulating the behaviour of the particles using some simple assumptions (such as Stokes law) about their interaction with gas, it is easier to use discrete particle simulation procedures such as molecular dynamics or event-driven simulation. If one is interested in treating exactly the complex interaction between them, a technique like the lattice Boltzmann simulation would then be useful.

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