ELSEVIER

Physica A 275 (2000) 483–504

PHYSICA A

www.elsevier.com/locate/physa

Asymptotic solution of the Boltzmann equation for the shear flow of smooth inelastic disks

V. Kumaran

Department of Chemical Engineering, Indian Institute of Science, Bangalore 560 012, India

Received 12 November 1998; received in revised form 15 March 1999

Abstract

The velocity distribution for a homogeneous shear flow of smooth nearly elastic disks is determined using a perturbation solution of the linearised Boltzmann equation. An expansion in the parameter $\varepsilon_I = (1-e)^{1/2}$ is used, where e is the coefficient of restitution. In the leading order approximation, inelastic effects are neglected and the distribution function is a Maxwell–Boltzmann distribution. The corrections to the distribution function due to inelasticity are determined using an expansion in the eigenfunctions of the linearised Boltzmann operator, which form a complete and orthogonal basis set. A normal form reduction is effected to obtain first-order differential equations for the coefficients of the eigenfunctions, and these are solved analytically subject to a set of simple model boundary conditions. The $O(\varepsilon_I)$ and $O(\varepsilon_I^2)$ corrections to the distribution function are calculated for both infinite and bounded shear flows. For a homogeneous shear flow, the results for the $O(\varepsilon_I)$ and $O(\varepsilon_I^2)$ corrections to the distribution function are different from those obtained earlier by the moment expansion method and the Chapman–Enskog procedure, but the numerical value of the corrections are small for the second moments of the velocity distribution, and the numerical results obtained by the different procedures are very close to each other. The variation in the distribution function due to the presence of a solid boundary is analysed, and it is shown that there is an $O(\varepsilon_I^2)$ correction to the density and an $O(\varepsilon_I)$ correction to the mean velocity due to the presence of a wall. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Granular materials; Shear flow; Boltzmann equation

PACS: 05.20Dd; 46.10+z; 62.90+k

1. Introduction

The shear flow of an inelastic granular material has been a widely studied problem in the field of granular flows. Many unusual phenomena, such as the formation of dense clusters and inhomogeneities [1–5] occur due to the inelasticity of the collisions

PII: S0378-4371(99)00263-0

between the particles in the flow. There have been two distinct types of theoretical studies on these systems, i.e. the continuum theories for slow flows of granular materials [6] and the kinetic theories for rapid flows [7–13]. The kinetic theories make use of the similarity between the dynamics of the grains in the granular flow and the molecules of a hard sphere gas. However, there are significant differences between the two systems. The particles in a granular medium are macroscopic objects, and the length scales of the flow are typically of the same magnitude as the mean free path. The diameters of the particles could also be of the same magnitude as the mean free path, thereby resulting in correlated collisions and invalidating the molecular chaos approximation. In addition, the interactions between the particles are inelastic and do not conserve energy. Motion can be sustained only in the presence of a continuous source of energy, for example, due to shearing at the boundaries. Consequently, the 'granular temperature', which is proportional to the mean square of the velocity fluctuations in the system, is related to the driving and dissipation of energy, in contrast to molecular gases where the temperature is a thermodynamic variable. There are gradients in the density and temperature of a granular material even under steady conditions, because the resultant energy flux is necessary to sustain the motion of the particles.

The velocity distribution function for hard sphere gases is derived from the Boltzmann equation, which is a non-linear integro-differential equation [14]. At steady state, it can be shown that the solution for this equation is the Maxwell–Boltzmann distribution. Transport equations are derived assuming that the deviation from the Maxwell–Boltzmann distribution is small, and using a perturbation solution of the Boltzmann equation. For gases close to equilibrium, there is an additional assumption that the variation in the distribution function is only due to the variation in the density, mean velocity and temperature. This assumption is valid only when the gas is close to an underlying equilibrium state, and the length scale of variation of macroscopic properties is large compared to the mean free path of the gas. In kinetic theories for granular materials, it is usually assumed that the distribution function is close to a Maxwell–Boltzmann distribution. This is valid when the coefficient of restitution is close to 1, and the length scale of variation of the properties is large compared to the mean free path of the particles. In addition, the assumption of molecular chaos, that there is no correlation in the pre-collisional velocities of a pair of particles, is also usually made; this assumption is valid only in dilute regimes. Despite these difficulties, kinetic theories have been fairly successful in predicting the behaviour of granular materials. There have been two variants of the classical kinetic theory that have been used, and both of these employ a perturbation expansion about the limit of elastic collisions. The first is a modification of the Chapman–Enskog approximation where the variation in the properties of the material are considered to be due to the spatial variation of the density, mean velocity and temperature [8], while the second is a moment expansion method, where a form of the expansion for the distribution function is assumed, and the unknown coefficients in the expansion are determined from a set of equations for the moments of the distribution function. A modification of the Chapman–Enskog procedure was used by Sela et al. [12], where a form of the perturbation to the distribution function is assumed based on the symmetries of the inhomogeneous terms in the equations for the correction to the distribution function. All these methods give results for the granular temperature and the anisotropy in a homogeneous shear flow that are in close agreement.

As mentioned earlier, the rapid flow of a granular material requires continuous driving. In a shear flow, this is achieved by the presence of solid boundaries which drive the material. Due to the boundary effects, one would expect significant variations in the distribution function. In the analysis of Jenkins and Richman [11], the effect of the wall has been included using flux conditions for the momentum and energy, which are derived using a microscopic averaging over the interaction of the particles with the boundary. In this averaging, the distribution function of the particles is assumed to be a Maxwell–Boltzmann distribution. Spatial variations have also been incorporated into hydrodynamic equations derived for granular materials, which provide effective equations for the transport of mass, momentum and energy in the medium.

An asymptotic analysis is used here to determine the perturbation to the distribution function in the shear flow of a granular material in the limit $(1-e) \ll 1$, where e is the coefficient of restitution. In addition, it is assumed that there is no correlation in the precollisional velocities of a pair of particles, and that the material is sufficiently dilute in that the radius of a particle is small compared to the mean free path. The latter approximation can be improved using the Enskog procedure for dense gases. However, this differs from previous studies that the mean free path is not considered to be small compared to the macroscopic length scale of the variation of properties. In addition, it is not assumed that the properties of the system can be described by conservation equations for the mass, momentum and energy, but other forms of corrections to the distribution function are also incorporated.

In order to place the present analysis in perspective, it is useful to compare the spatial variation of a granular material and the evolution of a rarefied gas to its final equilibrium state. In a gas at very low densities, there is the 'transition' regime, where the difference in the scale length between the macroscopic and microscopic scales is not large, and the Navier–Stokes equations are not sufficient to describe the state of the gas. In this case, a larger number of variables are required to completely specify the state. Very close to equilibrium in denser gases, there is the kinetic regime where the state is completely specified by the equations for the density, momentum and energy. In a granular material, there is continuous driving of the material at solid boundaries, and one would expect that there are domains near the boundaries of length scale comparable to the mean free path of the particles where the continuum equations are not sufficient to describe the variation of properties. When the distance from the boundaries is sufficiently large, it is expected that the continuum equations would be applicable if the coefficient of restitution is close to 1. In addition, there could be situations where the density is sufficiently low that the distance between boundaries is of the same magnitude as the mean free path. The purpose of the present analysis is to develop techniques that could be used in situations where the continuum mass, momentum and energy equations are not sufficient to describe the dynamics of the system, but additional variables are required.

The method of solution involves an expansion of the perturbation to the distribution function in the eigenfunctions of the linearised Boltzmann operator. It can be shown [15] that these form a complete and orthogonal function space, and are an appropriate basis set for the expansion. The eigenfunctions for the linearised Boltzmann operator for the hard sphere or hard disk model cannot be determined analytically, and is necessary to determine them numerically using an expansion in an appropriate function space. Here, an expansion in a function space consisting of Hermite polynomials is used. There are four eigenfunctions (in two dimensions), the mass, momenta and energy, whose corresponding eigenvalues are zero, while all the other eigenvalues are negative. This implies that the mass, momentum and energy remain unchanged, while any transient perturbations to the other eigenfunctions decay exponentially in time in a spatially homogeneous system. There are fluxes of mass, momentum and energy in a spatially non-uniform system, but the perturbations to the other eigenfunctions can be neglected if the length scale of variation of the properties is large compared to the mean free path of the gas.

A sheared granular material is not an equilibrium system, and so there are perturbations to the distribution function due to the inelastic nature of the collisions. An important difference between a non-equilibrium gas and a granular material is that the presence of inelastic collisions results in an additional inhomogeneous correction to the linearised Boltzmann equation, in addition to the corrections due to the gradients in the density, mean velocity and temperature. Consequently, the Chapman–Enskog approximation is not strictly valid, and a more general form has to be assumed for the perturbation to the distribution function even in a homogeneous flow. This analysis is carried out in the next section, where a hierarchy of equations is derived systematically for the correction to the distribution function in the limit $\varepsilon_I = (1-e)^{1/2} \ll 1$. The results are compared with earlier studies using the Chapman–Enskog procedure and the moment expansion procedure for homogeneous shear flows in Section 3, and good agreement is observed. The analysis is extended to the spatial variations in properties near a solid wall in Section 4. The equations for the correction to the distribution function are reduced to a normal form, and solutions for these are obtained analytically. The results indicate that there is a variation of $\mathrm{O}(\varepsilon_I)$ in the mean velocity and $\mathrm{O}(\varepsilon_I^2)$ in the density and the anisotropy in the distribution function due to the presence of a wall.

2. General formulation

A two-dimensional sheared suspension of inelastic particles is considered, where the X and Y are the flow and gradient directions, respectively. The mean velocity $U_m(y)$ in the X direction is a function of the Y coordinate. The dynamics of the system is described using a distribution function $f(\mathbf{X}, \mathbf{U}, t^*)$, where $f(\mathbf{X}, \mathbf{U}, t^*)\,\mathrm{d}\mathbf{X}\,\mathrm{d}\mathbf{U}$ is the number of particles in the differential volume d$\mathbf{X}$ about $\mathbf{X}$ in real space and in the differential volume d$\mathbf{U}$ about $\mathbf{U}$ in velocity space, and t^* is the dimensional time variable. Here, the 'fluctuating velocity' $\mathbf{U}$ is the difference between the actual particle velocity and the mean velocity at the particle position. The steady-state Boltzmann equation, which is the conservation equation for the distribution function, for a system homogeneous in all directions except the Y direction, is

$$\frac{\partial U_y f}{\partial Y} - \Gamma \frac{\partial U_y f}{\partial U_x} = \frac{\partial_c f}{\partial t^*}, \tag{1}$$

where Γ is the strain rate $(\mathrm{d}U_m/\mathrm{d}Y)$. The first term on the left represents the rate of change of the distribution function due to the motion in real space, while the second term is the rate of change of the distribution function due to the variation of the mean velocity with position [14]. The term on the right is the rate of change of the distribution function due to particle interactions.

It is useful to define scaled distance and velocity coordinates at the outset, and to identify the dimensionless small parameter used for the perturbation expansion. As stated earlier, it is assumed that the inelasticity of the particles is small, so the distribution function is close to the Maxwell–Boltzmann distribution. In this case, the velocities are scaled by $T^{1/2}$, where T is the 'temperature' (the mass of the particles is set equal to 1). The temperature in this case is not a thermodynamically prescribed quantity, however, but is determined by a balance between the source of energy due to shear and the dissipation due to inelastic collisions. The characteristic time scale is the inverse of the collision frequency, $(ndT^{1/2})^{-1}$, where n is the number of particles per unit area and d is the particle diameter. The rate of dissipation of energy per unit area due to inelastic collisions scales as $(\varepsilon_I^2 n^2 d T^{3/2})$, where $\varepsilon_I = (1-e)^{1/2}$ is a small parameter. The rate of production of energy per unit area due to shear is $\mathrm{O}((T^{1/2}/d)\Gamma^2)$, since the viscosity of a two-dimensional gas (divided by the mass of a particle) is $\mathrm{O}(T^{1/2}/d)$. Consequently, the shear rate Γ is $\mathrm{O}(\varepsilon_I n d T^{1/2})$ to achieve a balance between the source and dissipation of energy. The scaled velocity, distance, time and strain rate are defined as $\mathbf{u} = (\mathbf{U}/T^{1/2})$, $\mathbf{x} = (\mathbf{X}nd)$, $t = (t^* n d T^{1/2})$ and $\varepsilon_I \gamma = (\Gamma / n d T^{1/2})$, and the scaled Boltzmann equation is

$$\frac{\partial u_y f}{\partial y} - \varepsilon_I \gamma \frac{\partial u_y f}{\partial u_x} = \frac{\partial_c f}{\partial t}. \tag{2}$$

The dimensionless collision integral $(\partial_c f/\partial t)$ assumes the following form for the case where simultaneous interactions between three or more particles are neglected (binary collision approximation) and correlations in the pre collisional velocities of the particles are neglected (molecular chaos assumption)

$$\begin{aligned}\frac{\partial_c f}{\partial t} &= \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{u}^* \left[\frac{1}{e^2} f(\mathbf{x},\mathbf{u}') f(\mathbf{x}^*,\mathbf{u}^{*\prime}) - f(\mathbf{x},\mathbf{u}) f(\mathbf{x}^*,\mathbf{u}^*) \right] (\mathbf{w}.\mathbf{k}) \\ &= \mathscr{C}[f(\mathbf{x},\mathbf{u})], \end{aligned} \tag{3}$$

where $\mathbf{k}$ is the line joining the centres of the particles at collision, and the integrals are carried out over $\mathbf{w}.\mathbf{k} \geqslant 0$ [12]. The particle diameter is considered to be small compared to the mean free path, so that the difference in the positions of particles at collision is neglected ($\mathbf{x} = \mathbf{x}^*$). This approximation can be systematically improved using the Enskog procedure. The first term on the right-hand side of (3) is the rate of accumulation of particles at $(\mathbf{x}, \mathbf{u})$ in phase space due to a collision between particles at $(\mathbf{x}, \mathbf{u}')$ and $(\mathbf{x}^*, \mathbf{u}^{*\prime})$ such that the particle at $\mathbf{x}$ has a final velocity $\mathbf{u}$. The factor $(1/e^2)$ in the first term on the right-hand side accounts for the decrease in normal relative velocity by a factor e during a collision, and the contraction of a differential volume in phase space due to the decrease in the normal relative velocity. The second term is the rate of depletion of particles at $(\mathbf{x}, \mathbf{u})$ due to collisions involving a particle with velocity $\mathbf{u}$.

The velocities of the particles can be expressed in terms of a centre of mass velocity $\mathbf{v}$ and a relative velocity $\mathbf{w}$

$$\mathbf{v} = \frac{\mathbf{u} + \mathbf{u}^*}{2}, \qquad \mathbf{w} = \mathbf{u} - \mathbf{u}^* . \tag{4}$$

The centre of mass velocity $\mathbf{v}$ remains unchanged in a collision, while the velocity difference after the collision $\mathbf{w}_f$ is related to that before the collision $\mathbf{w}_i$ by

$$\mathbf{w}_f = (\mathbf{I} - (1 + e)\mathbf{kk}).\mathbf{w}_i , \tag{5}$$

where $\mathbf{I}$ is the second-order identity tensor, and e is the coefficient of restitution for particle–particle collisions. The velocities before $(\mathbf{u}_i, \mathbf{u}_i^*)$ and after $(\mathbf{u}_f, \mathbf{u}_f^*)$ a collision of a pair of particles are related by

$$\mathbf{u}_f = \mathbf{u}_i - (1 + e)\mathbf{kk}.(\mathbf{u}_i - \mathbf{u}_i^*) , \tag{6}$$

$$\mathbf{u}_f^* = \mathbf{u}_i^* + (1 + e)\mathbf{kk}.(\mathbf{u}_i - \mathbf{u}_i^*) . \tag{7}$$

In this analysis, the limit $\varepsilon_I = (1-e)^{1/2} \ll 1$ is considered, where the average dissipation of energy during a collision is small compared to the energy of the particle. In this case, it is useful to separate the collision integral into two parts

$$\mathscr{C}[f(\mathbf{x}, \mathbf{u})] = \mathscr{C}_e[f(\mathbf{x}, \mathbf{u})] + \mathscr{C}_i[f(\mathbf{x}, \mathbf{u})] , \tag{8}$$

where $\mathscr{C}_e$ is the collision operator for elastic particles

$$\mathscr{C}_e = \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{u}^* \, [f(\mathbf{u}'', \mathbf{x}) f(\mathbf{u}^{*\prime\prime}, \mathbf{x}) - f(\mathbf{u}, \mathbf{x}) f(\mathbf{u}^*, \mathbf{x})](\mathbf{w}.\mathbf{k}) \tag{9}$$

and $\mathscr{C}_i$ is the correction to the collision operator due to the inelasticity of the particles

$$\mathscr{C}_i = \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{u}^* \left[\frac{1}{e^2} f(\mathbf{u}', \mathbf{x}) f(\mathbf{u}^{*\prime}, \mathbf{x}) - f(\mathbf{u}'', \mathbf{x}) f(\mathbf{u}^{*\prime\prime}, \mathbf{x}) \right] (\mathbf{w}.\mathbf{k}) . \tag{10}$$

In Eqs. (9) and (10), the velocities $\mathbf{u}''$ and $\mathbf{u}^{*\prime\prime}$ are the velocities of the particles before an *elastic* collision such that one of the particles has a final velocity $\mathbf{u}$. It can easily be verified that $\mathscr{C}_i$ is $O(\varepsilon_I)$ smaller than $\mathscr{C}_e$ for the present case.

In the absence of dissipation due to inelastic collisions, the solution for the distribution function is the Maxwell–Boltzmann distribution $F(\mathbf{u})$ for a two-dimensional system

$$F(\mathbf{u}) = \frac{1}{(2\pi)} \exp\left(-\frac{u^2}{2} \right) . \tag{11}$$

This distribution function identically satisfies the leading order Boltzmann equation in the absence of non-equilibrium effects

$$\mathscr{C}_e[F(\mathbf{u})] = 0\,. \tag{12}$$

The correction to the distribution function due to inelastic effects is incorporated using an expansion of the form

$$f(\mathbf{x},\mathbf{u}) = F(\mathbf{u})[1 + \varepsilon_I \Phi^{(1)}(\mathbf{x},\mathbf{u}) + \varepsilon_I^2 \Phi^{(2)}(\mathbf{x},\mathbf{u}) + \cdots]\,. \tag{13}$$

The above expansion is inserted into the elastic part of the collision operator to obtain

$$\mathscr{C}_e[f(\mathbf{x},\mathbf{u})] = \varepsilon_I F(\mathbf{u})\mathscr{L}[\Phi^{(1)}] + \varepsilon_I^2 F(\mathbf{u})(\mathscr{L}[\Phi^{(2)}] + \mathscr{N}^{(2)}[\Phi^{(1)}])\,. \tag{14}$$

The operator $\mathscr{L}$ is a linear collision operator in which the collision is considered to be elastic

$$\mathscr{L}[\Phi] = \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{u}^*\, F(\mathbf{u}^*)[\Phi(\mathbf{x},\mathbf{u}'') + \Phi(\mathbf{x},\mathbf{u}^{*\prime\prime}) - \Phi(\mathbf{x},\mathbf{u}) - \Phi(\mathbf{x},\mathbf{u}^*)]\mathbf{w}.\mathbf{k} \tag{15}$$

and $\mathscr{N}^{(2)}[\Phi^{(1)}]$ is

$$\begin{aligned}\mathscr{N}^{(2)}[\Phi^{(1)}] = \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{u}^*\, & F(\mathbf{u})F(\mathbf{u}^*)(\Phi^{(1)}(\mathbf{x},\mathbf{u}'')\Phi^{(1)}(\mathbf{x},\mathbf{u}^{*\prime\prime}) \\ & - \Phi^{(1)}(\mathbf{x},\mathbf{u})\Phi^{(1)}(\mathbf{x},\mathbf{u}^*))\mathbf{w}.\mathbf{k}\,.\end{aligned} \tag{16}$$

The inhomogeneous term (10) on the right-hand side of (8) due to the inelastic part of the collision operator is determined using an expansion in the parameter ε_I.

$$\mathscr{C}_i = \varepsilon_I \mathscr{C}_i^{(1)} + \varepsilon_I^2 \mathscr{C}_i^{(2)}\,. \tag{17}$$

It is convenient to define the distribution functions $F(\mathbf{v})$ and $F(\mathbf{w})$ functions for the velocity of the centre of mass and the velocity difference

$$\begin{aligned} F_v(\mathbf{v}) &= \frac{1}{\pi}\exp(-v^2)\,, \\ F_w(\mathbf{w}) &= \frac{1}{4\pi}\exp\left(-\frac{w^2}{4}\right).\end{aligned} \tag{18}$$

The O(ε_I) contribution to $\mathscr{C}_i$ is determined using a Taylor series expansion and retaining terms correct to O(ε_I^2)

$$\begin{aligned} F_w(\mathbf{w}') &= F_w(\mathbf{w}'') + (\mathbf{w}' - \mathbf{w}'').\nabla_{w''} F_w(\mathbf{w}'') \\ &= \left[-\varepsilon_I^2 \frac{(\mathbf{w}.\mathbf{k})^2}{2}\right] F_w(\mathbf{w})\,.\end{aligned} \tag{19}$$

Using the above relations, it can easily be verified that the O(1) contribution to $\mathscr{C}_i$ is zero, while the O(ε_I) and O(ε_I^2) contributions are

$$\mathscr{C}_i^{(1)} = 0\,, \tag{20}$$

$$\mathscr{C}_i^{(2)} = \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{u}^*\, F(\mathbf{v})F(\mathbf{w})\left(2 - \frac{(\mathbf{w}.\mathbf{k})^2}{2}\right). \tag{21}$$

Inserting the above into the Boltzmann equation, an equation of the following form is obtained for the correction $\Phi^{(n)}$:

$$\frac{\partial u_y F\Phi^{(n)}}{\partial y} - F(\mathbf{u})\mathscr{L}[\Phi^{(n)}] = \mathscr{C}_i^{(n)}[\Phi^{(n-1)},\dots,\Phi^{(1)}] + \mathscr{N}^{(n)}[\Phi^{(n-1)},\dots,\Phi^{(1)}] + \gamma\frac{\partial(u_y F\Phi^{(n-1)})}{\partial u_x} . \tag{22}$$

The approach used for solving an equation of the form (22) for $\Phi^{(n)}$ is as follows. It can be shown that the operator $\mathscr{L}[\phi]$ is self-adjoint when the inner product in velocity space is defined with the weighting function $F(\mathbf{u})$ [15,14], i.e. for any real functions $g(\mathbf{u})$ and $h(\mathbf{u})$

$$\int \mathrm{d}\mathbf{u}\, F(\mathbf{u})h(\mathbf{u})\mathscr{L}[g(\mathbf{u})] = \int \mathrm{d}\mathbf{u}\, F(\mathbf{u})g(\mathbf{u})\mathscr{L}[h(\mathbf{u})] . \tag{23}$$

In addition, it can also be shown that the eigenvalues λ_J of the equation

$$\mathscr{L}[\phi_J(\mathbf{u})] = \lambda_J\phi_J(\mathbf{u}) \tag{24}$$

are real, discrete and satisfy the condition

$$\lambda_J \leqslant 0 \tag{25}$$

while the eigenfunctions form an orthogonal basis set. There are four eigenvalues (in two dimensions) which are identically zero, and the eigenfunctions corresponding to these are the mass, two components of the momenta and the energy of the particles which are conserved in collisions. All other eigenvalues are less than zero.

A solution for the linear equation (22) can be obtained as follows. The correction to the distribution function $\Phi^{(n)}$ is expressed as an expansion in the eigenfunctions ϕ_J of Eq. (24)

$$\Phi^{(n)} = \sum_J A_J^{(n)}(\mathbf{x})\phi_J(\mathbf{u}) . \tag{26}$$

Similarly, the inhomogeneous term on the right-hand side of (22) is also expressed as an expansion in the eigenfunctions ϕ_J

$$\mathscr{C}_i^{(n)} + \mathscr{N}^{(n)} = \sum_J B_J^{(n)}(\mathbf{x})\phi_J(\mathbf{u}) . \tag{27}$$

These Boltzmann equation (22) now reduces to an equation for the coefficients $A_J^{(n)}$

$$M_{IJ}\frac{\partial A_J^{(n)}}{\partial y} - \Lambda_{IJ}A_J^{(n)} = B_I^{(n)} + \gamma N_{IJ}A_J^{(n-1)} , \tag{28}$$

where the matrices M_{IJ} and T_{IJ} are

$$M_{IJ} = \int \mathrm{d}\mathbf{u}\, F(\mathbf{x},\mathbf{u})u_y\phi_I(\mathbf{u})\phi_J(\mathbf{u}) ,$$

$$N_{IJ} = \int \mathrm{d}\mathbf{u}\, \phi_I(\mathbf{u})\frac{\partial(Fu_y\phi_J(\mathbf{u}))}{\partial u_x} \tag{29}$$

and Λ_{IJ} is a diagonal matrix whose diagonal elements are the eigenvalues of $\mathscr{L}$. The above equation can be solved systematically for the corrections to the distribution function $\Phi^{(n)}$.

3. Homogeneous shear flow

Before proceeding to examine inhomogeneous flows, it is useful to first examine the results obtained by the present procedure for a homogeneous shear flow. In this case, the coefficients $A_J^{(n)}$ are independent of the coordinate y, and the equations for the coefficients $A_J^{(n)}$ reduce to

$$- \Lambda_{IJ} A_J^{(n)} = B_I^{(n)} + \gamma N_{IJ} A_J^{(n-1)} . \tag{30}$$

As noted earlier, the matrix Λ_{IJ} is a diagonal matrix, whose diagonal elements are the eigenvalues λ_J. There are four eigenvalues λ_{N-3} to λ_N which are identically zero, corresponding to the collisionally invariant eigenfunctions proportional to the mass, momentum and energy

$$\begin{aligned}
\phi_{N-3}(\mathbf{u}) &= 1 , \\
\phi_{N-2}(\mathbf{u}) &= u_x , \\
\phi_{N-1}(\mathbf{u}) &= u_y , \\
\phi_N(\mathbf{u}) &= u_x^2 + u_y^2 - 2 .
\end{aligned} \tag{31}$$

Therefore, Eq. (30) has solutions only if the right-hand side is zero for $I=(N-3)$ to N, and this provides the solvability condition for Eq. (30). These solvability conditions are used, at each order in perturbation theory, to determine the leading order and higher corrections to the strain rate γ. It can be shown from symmetry arguments [14] that the inhomogeneous terms in (30) corresponding to the eigenfunctions 1, u_x and u_y are identically zero due to the requirements of mass, momentum and energy conservation. However, since particle energies are not conserved in a collision, the inhomogeneous term corresponding to the energy eigenfunction $(u_x^2 + u_y^2 - 2)$ is zero only for certain values of the macroscopic shear rate γ, and this condition provides a solution for the shear rate γ.

The eigenfunctions for the linearised collision operator are difficult to obtain for a hard sphere interaction; the only case where the eigenfunctions are easily obtained is for a Maxwell molecule where the intermolecular potential decays as the inverse fifth power of the distance of separation [15]. In the present analysis, the eigenvalues and eigenfunctions are determined as expansions in a Hermite polynomial series for a two-dimensional system

$$\phi_I(\mathbf{u}) = \sum_{J=1}^{N} P_{IJ} \psi_J(\mathbf{u}) , \tag{32}$$

where ψ_J are the products of Hermite polynomials of the particle velocities in the horizontal and vertical directions. The choice of Hermite polynomials is appropriate because they form a basis set of orthogonal polynomials when the inner product is defined on the domain $-\infty$ to ∞ with $F(\mathbf{u})$ as the weighting factor. The eigenfunctions for the collision operator can be determined from the matrix G_{IJ}

$$G_{IJ} = \int d\mathbf{u}\, F(\mathbf{u})\psi_I(\mathbf{u})\mathscr{L}[\psi_J(\mathbf{u})] . \tag{33}$$

It can be easily shown that the matrix P_{IJ} in (32) contains as its rows the eigenvectors of the matrix G_{IJ}, and the eigenvalues λ_J in (24) are identical to the eigenvalues of the matrix G_{IJ}. Using (32), relation (24) reduces to

$$\sum_{K=1}^{N} P_{IK}\mathscr{L}[\psi_K(\mathbf{u})] = \sum_{J=1}^{N}\sum_{L=1}^{N} P_{JL}\psi_L\Lambda_{JI} , \tag{34}$$

where Λ_{JI} is a diagonal matrix whose diagonal terms are the eigenvalues λ_J. Multiplying both sides of the above equation by $\psi_M(\mathbf{u})F(\mathbf{u})$ and integrating over $\mathbf{u}$, the following relation is obtained:

$$\sum_{K=1}^{N} G_{MK}P_{KI}^{\mathrm{T}} = \sum_{J=1}^{N} P_{MJ}^{\mathrm{T}}\Lambda_{JI} . \tag{35}$$

Here, it is assumed that the basis functions are normalised, so that

$$\int d\mathbf{u}\, F(\mathbf{u})\psi_I(\mathbf{u})\psi_J(\mathbf{u}) = I_{IJ} , \tag{36}$$

where I_{IJ} is the identity matrix. The matrix M_{IJ}, required for determining the properties of inhomogeneous flows in the next section, is

$$\begin{aligned} M_{IJ} &= \int d\mathbf{u}\, F(\mathbf{u})u_y\phi_I(\mathbf{u})\phi_J(\mathbf{u}) \\ &= \sum_{K=1}^{N}\sum_{L=1}^{N} P_{IK}P_{JL}\int d\mathbf{u}\, F(\mathbf{u})u_y\psi_K(\mathbf{u})\psi_L(\mathbf{u}) \\ &= \sum_{K=1}^{N}\sum_{L=1}^{N} P_{IK}P_{JL}R_{KL} . \end{aligned} \tag{37}$$

The other transformations then follow as detailed above. Results were obtained for $N=16$ and 36 (corresponding to the first four and six Hermite polynomials in the horizontal and vertical directions). Eq. (35) then gives 16 or 36 eigenvalues and eigenvectors for the linear collision operator. As discussed earlier, four of the eigenvalues are zero, while all the others are negative. The eigenvalues λ_J for $N = 16$ and 36 are shown in Table 1. The eigenfunctions ϕ_J for $N = 16$ are shown in Table 2.

The corrections to the distribution functions, $\Phi^{(1)}$ and $\Phi^{(2)}$, obtained using the present solution procedure, are provided in Appendix A. The moments of the distribution function obtained using the present procedure are compared with the previous results in Table 1. It is seen that the second moments $\langle u_x u_y\rangle$, $\langle u_x^2\rangle$ and $\langle u_y^2\rangle$ do not show significant variation when the value of N is changed from 16 to 36, indicating that a basis

Table 1
The non-zero eigenvalues γ_I for $N = 16$ and 36. In addition to these, there are a total of six eigenvalues that are zero

$N = 16$	$N = 36$
	−9.071454
	−8.727477
	−8.517621
	−8.517621
	−8.212974
	−8.212974
	−8.057390
	−7.874723
	−7.600401
	−7.558564
−6.602094	−7.352401
−6.425145	−7.352401
−6.042078	−6.798699
−6.042078	−6.298625
−5.261972	−6.298625
−5.182199	−6.024768
−5.182199	−5.389357
−4.125816	−5.307500
−3.544907	−5.307500
−3.099046	−4.718563
−1.764485	−4.637607
−1.764485	−4.342166
	−4.342166
	−4.172728
	−3.732822
	−3.128074
	−2.916262
	−2.916262
	−2.837848
	−1.749309
	−1.658600
	−1.658600

set with 16 functions is sufficient to accurately capture the behaviour of the second moments of the distribution function. However, there is a significant variation in the behaviour of the fourth moments of about 10–12% when the number of basis functions is increased from 16 to 36, indicating that a larger set of basis functions is required to capture the behaviour of these moments.

The moments obtained using the present analysis is also compared with the earlier studies of Sela et al. [12] and Jenkins and Richman [9]. The second moments of the distribution function obtained by these studies are in good agreement, and it is useful to examine the reasons for this agreement. The inhomogeneous term in the equation for $\Phi^{(1)}$ is proportional to $u_x u_y$, and symmetry arguments indicate that the first correction should be of the form

$$\Phi^{(1)} = u_x u_y g(u) , \tag{38}$$

Table 2
The eigenfunctions ϕ_I for $N = 16$

$\psi_1 = 1.194979u_x u_y - 0.441904u_x^3 u_y - 0.441904u_x u_y^3 + 0.164079u_x^3 u_y^3$
$\psi_2 = 0.288675u_x^3 u_y - 0.288675u_x u_y^3$
$\psi_3 = -0.739354u_y + 0.537739u_x^2 u_y + 0.331975u_y^3 - 0.264770u_x^2 u_y^3$
$\psi_4 = -0.739354u_x + 0.331975u_x^3 + 0.537739u_x u_y^2 - 0.264770u_x^3 u_y^2$
$\psi_5 = 0.5 - 0.5u_x^2 - 0.5u_y^2 + 0.5u_x^2 u_y^2$
$\psi_6 = -0.313239u_y + 0.897686u_x^2 u_y - 0.080498u_y^3 - 0.114317u_x^2 u_y^3$
$\psi_7 = -0.313239u_x - 0.080498u_x^3 + 0.897686u_x u_y^2 - 0.114317u_x^3 u_y^2$
$\psi_8 = -0.838959u_x u_y + 0.285587u_x^3 u_y + 0.285587u_x u_y^3 - 0.024074u_x^3 u_y^3$
$\psi_9 = 0.5u_x^2 - 0.5u_y^2$
$\psi_{10} = -2.029327u_x u_y + 0.237682u_x^3 u_y + 0.237682u_x u_y^3 - 0.016622u_x^3 u_y^3$
$\psi_{11} = 1.450942u_y - 0.393694u_x^2 u_y - 0.365119u_y^3 + 0.012703u_x^2 u_y^3$
$\psi_{12} = 1.450942u_x - 0.365119u_x^3 - 0.393694u_x u_y^2 + 0.012703u_x^3 u_y^2$
$\psi_{13} = -1.0 + 0.5u_x^2 + 0.5u_y^2$
$\psi_{14} = u_x$
$\psi_{15} = u_y$
$\psi_{16} = 1.0$

where u is the magnitude of the particle velocity. In the Chapman–Enskog procedure, the function $g(u)$ is expanded in a complete basis set consisting of Sonine polynomials. In the present case, the solution is obtained using a basis set consisting of the eigenfunctions of the linearised Boltzmann operator. Due to the difference in the expansion procedure, the first correction is not exactly of the form (38). However, the terms that do not have the form (38) have small numerical coefficients, and this accounts for the close agreement between this and earlier results.

For a gas of elastic particles, similar arguments are used to determine the form of the second moment in the Chapman–Enskog procedure

$$\Phi^{(2)} = h_0(u) + h_1(u)u_x^2 + h_2(u)u_x u_y + h_4(u)u_x^4 \,. \tag{39}$$

This form of the correction is also inferred from symmetry arguments, and the functions $h_0 - h_4$ are determined using an expansion in a complete and orthogonal polynomial space. The moment expansion method makes a further approximation of the above expressions (as discussed in [12]) of assuming constant values for the functions g and h_i, which are roughly averages of the corresponding functions over the magnitude of the velocity u. In the present method, the symmetry arguments are not exploited, and the inhomogeneous terms are determined as an expansion in a set of basis functions which are the eigenfunctions of the linearised Boltzmann operator. However, as is evident from Appendix A, the resulting corrections to the distribution function satisfy the symmetry conditions to a good numerical approximation. This accounts for the good agreement between all three methods for predicting the second moments of the distribution functions. There are significant differences between the three methods for the fourth moments, however, because the terms that cannot be expressed in the form (38) and (39) provide larger contributions to the fourth moments. Even though the symmetry arguments are not exploited in the present calculations, there are significant advantages for calculating higher-order corrections as indicated below.

In the Chapman–Enskog procedure, the form of the correction to the distribution function is determined from the symmetries of the inhomogeneous term in the equation. While it is easy to deduce the form of the inhomogeneous term from symmetry arguments for the first correction, and it is manageable for the second correction, it becomes difficult to do so for the third and higher corrections. The present procedure has the advantage that linear operator (on the left-hand side of Eq. (28), for example) is unchanged for the higher corrections, while there is a change in the inhomogeneous terms on the right-hand side. Therefore, the coefficients $A_J^{(n)}$ on the left-hand side for the higher-order corrections are easily calculated once the inhomogeneous terms are determined. Moreover, it is easy to use the solvability conditions to determine the strain rate as a function of temperature. For example, for the homogeneous shear flow, the solvability condition for Eq. (30) requires that the inhomogeneous term on the right should be zero if the diagonal component of the matrix Λ_{IJ} is zero. This condition directly provides the leading order and higher-order corrections to the strain rate. In the Chapman–Enskog method, a more elaborate procedure is required to obtain the solvability conditions [12]. Another advantage is the ease with which the calculation can be extended to inhomogeneous shear flows, as illustrated below.

4. Inhomogeneous shear flow

In this section, the solution for the variation in the flow properties due to the presence of a moving boundary is examined. The flow geometry consists of a sheared granular material bounded by a moving wall at $Y = 0$, and the granular material flows in the region $Y < 0$ where X and Y are the coordinates in the flow and gradient directions, respectively. The reflection conditions at the wall in the present analysis are considered to be of the form

$$U_x' = e_t U_x'' + (1 - e_t) U_w U_y' = -e_n U_y'' \, , \tag{40}$$

where U_x' and U_y' are the dimensional velocities after the wall collision, U_x'' and U_y'' are the velocities before collision, U_w is the difference between the wall velocity and the mean velocity of the particles at the wall, and e_t and e_n are the tangential and normal coefficients of restitution.

Consistent with the asymptotic scheme used for the distribution function, an expansion about the limit of elastic collisions is used to determine the boundary conditions. The appropriate expansions for the coefficients of restitution are $e_t = (1 - a_t \varepsilon_I^2)$ and $e_n = (1 - a_n \varepsilon_I^2)$. The boundary condition for the velocity field at the wall, scaled by $\sqrt{T}$, is

$$u_x' = (1 - a_t \varepsilon_I^2) u_x'' + \varepsilon_I u_w, \qquad u_y' = -(1 - a_t \varepsilon_I^2) u_y'' \, , \tag{41}$$

where $u_w = U_w/(a_t \varepsilon_I \sqrt{T}))$, and (u_x'', u_y'') and (u_x', u_y') are the scaled particle velocities before and after the collision with the wall. Note that the scaled velocity u_w has to be O(1) for the scaled shear rate in the granular medium to be O(ε_I).

The distribution function is separated into two components

$$f(y, \mathbf{u}) = f_u(\mathbf{u}) + f_n(y, \mathbf{u}), \tag{42}$$

where $f_u(\mathbf{u})$ is the spatially uniform component determined in the previous section, and $f_n(y, \mathbf{u})$ is the correction to the spatially uniform distribution function due to the presence of the solid boundary. The non-uniform component $f_n(y, \mathbf{u})$ is expressed as an expansion in the parameter ε_I

$$f_n(y, \mathbf{u}) = F(\mathbf{u})(1 + \varepsilon_I \Phi_n^{(1)}(y, \mathbf{u}) + \varepsilon_I^2 \Phi_n^{(2)}(y, \mathbf{u}) + \cdots). \tag{43}$$

The Boltzmann equation for the functions $\Phi_n^{(1)}$ and $\Phi_n^{(2)}$ are

$$\partial_y (u_y F(\mathbf{u}) \Phi_n^{(1)}(y, \mathbf{u})) - \mathscr{L}[F(\mathbf{u}) \Phi_n^{(1)}(y, \mathbf{u})] = \mathscr{N}_n^{(1)}(y, \mathbf{u}), \tag{44}$$

$$\partial_y (u_y F(\mathbf{u}) \Phi_n^{(2)}(y, \mathbf{u})) - \mathscr{L}[F(\mathbf{u}) \Phi_n^{(2)}(y, \mathbf{u})] = \mathscr{N}_n^{(2)}(y, \mathbf{u}), \tag{45}$$

where

$$\mathscr{N}_n^{(1)}(y, \mathbf{u}) = 0 \tag{46}$$

and

$$\begin{aligned} \mathscr{N}_n^{(2)}(y, \mathbf{u}) = \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{u}^* \, F(\mathbf{u}) F(\mathbf{u}^*) (&\Phi_n^{(1)}(\mathbf{u}'') \Phi^{(1)}(\mathbf{u}^{*\prime\prime}) + \Phi^{(1)}(\mathbf{u}'') \Phi_n^{(1)}(\mathbf{u}^{*\prime\prime}) \\ &+ \Phi_n^{(1)}(\mathbf{u}'') \Phi_n^{(1)}(\mathbf{u}^{*\prime\prime}) - \Phi_n^{(1)}(\mathbf{u}) \Phi^{(1)}(\mathbf{u}^*) - \Phi^{(1)}(\mathbf{u}) \Phi_n^{(1)}(\mathbf{u}^*) \qquad (47) \\ &- \Phi_n^{(1)}(\mathbf{u}) \Phi_n^{(1)}(\mathbf{u}^*)) \mathbf{w}.\mathbf{k} \\ &+ \gamma \frac{\partial u_y F(\mathbf{u}) \Phi_n^{(1)}(y, \mathbf{u})}{\partial u_x}. \qquad (48) \end{aligned}$$

A normal form reduction for the non-equilibrium correction to the distribution function can be effected as follows. The corrections to the distribution function are expressed as

$$\Phi_n^{(n)} = \sum_I C_I^{(n)}(y) \phi_I(\mathbf{u}), \tag{49}$$

where $\phi_I(\mathbf{u})$ are the eigenfunctions of the linearised Boltzmann equation. The above expansion is inserted into the equations for the distribution function (44) and (45) to obtain

$$M_{IJ} \partial_y C_J^{(n)} - \Lambda_{IJ} C_J^{(n)} = D_I^{(n)}(y), \tag{50}$$

Table 3
The $O(\varepsilon_I^2)$ corrections to the moments $\langle\rho(u_x^2-u_y^2)\rangle$, $\langle\rho u_x^4\rangle$, $\langle\rho u_y^4\rangle$ and $\langle\rho u_x^2 u_y^2\rangle$ for an infinite shear flow calculated using $N=16$ and 36. Also, provided are the values of Sela et al. [12] (SKN) and Jenkins and Richman [9] (JR)

	$N=16$	$N=36$	SGN	JR
γ_0	3.50928	3.50693	3.5084	3.5449
$\langle u_x u_y\rangle/\varepsilon_I$	-1.0183	-1.01083	-1.0103	-1.0000
$\langle u_x^3 u_y\rangle/\varepsilon_I$	-2.78943	-2.78008	-3.0000	-3.0000
$\langle u_x u_y^3\rangle/\varepsilon_I$	-2.78943	-2.78008	-3.0000	-3.0000
$(\langle u_x^2\rangle-1)/\varepsilon_I^2$	1.0000	1.04014	1.048	1.0000
$(\langle u_y^2\rangle-1)/\varepsilon_I^2$	-1.0000	-1.04014	-1.048	-1.0000
$(\langle u_x^4\rangle-3)/\varepsilon_I^2$	6.0000	5.23346		
$(\langle u_y^4\rangle-3)/\varepsilon_I^2$	-6.0000	-5.23346		
$(\langle u_x^2 u_y^2\rangle-1)/\varepsilon_I^2$	2.22569	2.33637		

Table 4
The non-zero eigenfunctions σ_I for the spatially varying granular medium for $N=16$. In addition to these, there are six zero eigenvalues

$N=16$
± 5.978372
± 4.823394
± 2.756686
± 2.360328
± 1.878464

where

$$D_I^{(n)} = \int \mathrm{d}\mathbf{u}\, F(\mathbf{u})\mathcal{N}^{(n)}(y,\mathbf{u})\phi_I(\mathbf{u})\,, \tag{51}$$

$$M_{IJ} = \int \mathrm{d}\mathbf{u}\, F(\mathbf{u})u_y\phi_I(\mathbf{u})\phi_J(\mathbf{u}) \tag{52}$$

and Λ_{IJ} is the diagonal matrix of the eigenvalues of the linearised collision operator. Eq. (50) is multiplied by the inverse of the matrix M_{IJ} to obtain

$$\partial_y C_J^{(n)} - S_{IJ}C_J^{(n)} = M_{IJ}^{-1}D_J^{(n)}(y)\,, \tag{53}$$

where

$$S_{IJ} = \sum_K M_{IK}^{-1}\Lambda_{KJ}\,. \tag{54}$$

The matrix S_{IJ} has six eigenvalues that are zero, while the other eigenvalues occur in pairs of equal magnitude and opposite sign, $\sigma_2=-\sigma_1$, $\sigma_4=-\sigma_3,\ldots$ as shown in Table 4. The eigenfunctions, shown in Table 5, occur in pairs $\psi_2(u_x,u_y)=\psi_1(u_x,-u_y)$,

$\psi_4(u_x, u_y) = -\psi_3(u_x, -u_y), \ldots$. A normal form reduction is effected using the transformation

$$S_{IJ} = \sum_{K=1}^{N} \sum_{L=1}^{N} Q_{IK} \Sigma_{KL} Q_{LJ}^{-1} , \tag{55}$$

where Σ_{KL} is a matrix which has been reduced to a Jordan canonical form which consists of a $(N-6) \times (N-6)$ block diagonal corresponding to the non-zero eigenvalues, while the other rows and columns have non-diagonal terms, but the diagonal elements are zero since the eigenvalues are zero. Q_{IJ} is the corresponding matrix of eigenfunctions. Using this, Eq. (53) can be written as

$$\partial_y C_I^{*(n)} - \Sigma_{IJ} C_J^{*(n)} = D_I^{*(n)}(y) , \tag{56}$$

where

$$C_I^{*(n)} = \sum_{K=1}^{N} Q_{IK}^{-1} C_K^{(n)} , \tag{57}$$

$$D_I^{*(n)} = \sum_{K=1}^{N} \sum_{L=1}^{N} Q_{IK}^{-1} R_{KL}^{-1} D_L^{(n)} . \tag{58}$$

The coefficients for the eigenfunctions corresponding to non-zero eigenvalues are determined directly from (56). The requirement that the perturbations should decay into the medium implies that only the eigenfunctions with positive eigenvalues are retained in the expansion. The coefficients for the eigenfunctions with zero eigenvalues are then determined by solving the corresponding differential equation with known values for the other coefficients.

The boundary condition for the coefficients $C_I^{(n)}$ are determined from the condition that the flux of a moment of the distribution function at the interface, $\langle u_y \phi_I \rangle$, at the surface, is equal to the collisional rate of change of the moment due to collisions with the wall (per unit area of the wall)

$$\int_{-\infty}^{\infty} \mathrm{d}u_x \int_{-\infty}^{\infty} \mathrm{d}u_y \, u_y \phi_I(\mathbf{u}) f(0, \mathbf{u}) = \frac{\partial_w (f(\mathbf{u}) \phi_I(\mathbf{u}))}{\partial t} , \tag{59}$$

where

$$\frac{\partial_w (f(\mathbf{u}) \phi_I(\mathbf{u}))}{\partial t} = \int_{-\infty}^{\infty} \mathrm{d}u_x \int_{0}^{\infty} \mathrm{d}u_y \, u_y f(\mathbf{u}) (\phi_I(\mathbf{u}') - \phi_I(\mathbf{u}'')) , \tag{60}$$

where $\mathbf{u}'$ and $\mathbf{u}''$ are related by Eq. (41). The left-hand side of Eq. (43) is related to the coefficients $C_I^{*(n)}$ is as follows:

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{d}u_x \int_{-\infty}^{\infty} \mathrm{d}u_y \, u_y \phi_I(\mathbf{u}) f(0, \mathbf{u}) = & \int_{-\infty}^{\infty} \mathrm{d}u_x \int_{-\infty}^{\infty} \mathrm{d}u_y \, u_y F(\mathbf{u}) \phi_I(\mathbf{u}) \\ & + \sum_{J=1}^{N} M_{IJ} (\varepsilon_I (A_J^{(1)} + C_J^{(1)}(0)) \\ & + \varepsilon_I^2 (A_J^{(2)} + C_J^{(2)}(0)) + \cdots) . \end{aligned} \tag{61}$$

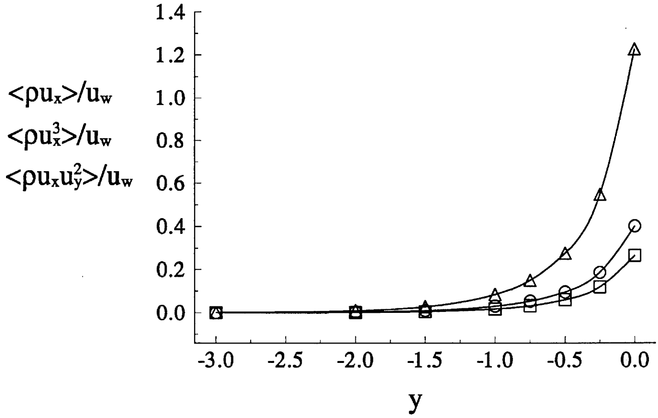

Fig. 1. The variation in the $O(\varepsilon_I)$ correction to the moments $\langle \rho u_x^3 \rangle / u_w$ and $\langle \rho u_x u_y^2 \rangle / u_w$ calculated using $N = 16$ as a function of the distance from the wall for a finite shear flow.

The leading term on the right-hand side of Eq. (60) is $O(\varepsilon_I)$ because the difference in the pre- and post-collisional velocities is $O(\varepsilon_I)$. Consequently, the $O(\varepsilon_I^n)$ contribution to the right-hand side of (60) only contains perturbations of $O(\varepsilon_I^{n-1})$, and Eq. (60) can be solved to provide the coefficients $C_I^{*(n)}$. The variation in the coefficients is determined using the differential equation (56). In addition, a valid solution of the differential equation (56) requires that the coefficients corresponding to the eigenfunctions with zero eigenvalues should be zero at the wall, otherwise the magnitude of these coefficients will increase linearly from the surface. This is ensured by choosing the pre-collisional velocities (u_x'', u_y'') and the post-collisional velocities (u_x', u_y') such that

$$
\begin{aligned}
& u_x'' = u_x(1 + a_t \varepsilon_I^2/2) - \varepsilon_I u_w/2 \quad u_x' = u_x(1 - a_t \varepsilon_I^2/2) + \varepsilon_I u_w/2 \,, \\
& u_y'' = u_y(1 + \varepsilon_I^2(a_n/2 - u_w^2/8)) \quad u_y' = (-1 + \varepsilon_I^2(a_n/2 + u_w^2/8)) \,.
\end{aligned} \tag{62}
$$

It is important to note that the above expressions for the velocity are correct only to $O(\varepsilon_I^2)$ and further refinement is necessary to obtain higher corrections. In addition, the solvability of the equations also requires that $\langle \rho u_x u_y \rangle = \varepsilon_I u_w / \sqrt{2\pi}$ at the wall. Since the stress $\langle \rho u_x u_y \rangle$ is the flux of a conserved quantity, i.e. the momentum flux in the x direction, this has to be a constant in the flow. The relation between this flux and the strain rate then determines the strain rate in the flow.

The calculations for the correction to the velocity distribution function have been carried out for $N = 16$ (see Appendix A and Table 5), and the results are as follows. The $O(\varepsilon_I)$ correction to the distribution function, $\Phi^{(1)}$, causes a change in the odd moments of the distribution function near the wall. These corrections are proportional to u_w, and the moments $\rho \langle u_x^3 \rangle / u_w$ and $\rho \langle u_x u_y^2 \rangle / u_w$ are shown as a function of y in Fig. 1. The third moments $\langle u_x^3 \rangle$ and $\rho \langle u_x u_y^2 \rangle$ are positive at the wall due to the acceleration of particles by the surface. The other third moments are zero.

Table 5
The eigenfunctions for the spatially varying granular medium for $N = 16$

$$
\begin{aligned}
\phi_1 &= 0.601891u_x - 0.295772u_x^3 - 1.039375u_xu_y + 0.483524u_x^3u_y \\
&\quad - 0.036237u_xu_y^2 + 0.107220u_x^3u_y^2 + 0.178630u_xu_y^3 - 0.105232u_x^3u_y^3 \\
\phi_2 &= -0.601891u_x + 0.295772u_x^3 - 1.039375u_xu_y + 0.483524u_x^3u_y \\
&\quad + 0.036237u_xu_y^2 - 0.107220u_x^3u_y^2 + 0.178630u_xu_y^3 - 0.105232u_x^3u_y^3 \\
\phi_3 &= -0.229640 + 0.539647u_x^2 + 0.636363u_y + 0.631501u_x^2u_y \\
&\quad - 0.080365u_y^2 - 0.229640u_x^2u_y^2 - 0.212121u_y^3 - 0.210500u_x^2u_y^3 \\
\phi_4 &= -0.229640 + 0.539647u_x^2 - 0.636363u_y - 0.631501u_x^2u_y \\
&\quad - 0.080365u_y^2 - 0.229640u_x^2u_y^2 + 0.212121u_y^3 + 0.210500u_x^2u_y^3 \\
\phi_5 &= 0.574911u_x - 0.103686u_x^3 + 0.779310u_xu_y - 0.079803u_x^3u_y \\
&\quad - 0.749503u_xu_y^2 + 0.161883u_x^3u_y^2 - 0.420438u_xu_y^3 + 0.080157u_x^3u_y^3 \\
\phi_6 &= 0.574911u_x - 0.103686u_x^3 - 0.779310u_xu_y + 0.079803u_x^3u_y \\
&\quad - 0.749503u_xu_y^2 + 0.161883u_x^3u_y^2 + 0.420438u_xu_y^3 - 0.080157u_x^3u_y^3 \\
\phi_7 &= -0.301001 + 0.150789u_x^2 + 1.815262u_y - 0.405054u_x^2u_y \\
&\quad + 0.451213u_y^2 - 0.301001u_x^2u_y^2 - 0.605087u_y^3 + 0.135018u_x^2u_y^3 \\
\phi_8 &= 0.301001 - 0.150789u_x^2 + 1.815262u_y - 0.405054u_x^2u_y \\
&\quad - 0.451213u_y^2 + 0.301001u_x^2u_y^2 - 0.605087u_y^3 + 0.135018u_x^2u_y^3 \\
\phi_9 &= -0.303259u_x - 0.015740u_x^3 - 1.088140u_xu_y + 0.124903u_x^3u_y \\
&\quad - 0.042200u_xu_y^2 + 0.130894u_x^3u_y^2 + 0.130628u_xu_y^3 + 0.035727u_x^3u_y^3 \\
\phi_{10} &= -0.303259u_x - 0.015740u_x^3 + 1.088140u_xu_y - 0.124903u_x^3u_y \\
&\quad - 0.042200u_xu_y^2 + 0.130894u_x^3u_y^2 - 0.130628u_xu_y^3 - 0.035727u_x^3u_y^3 \\
\phi_{11} &= 1 \\
\phi_{12} &= u_y \\
\phi_{13} &= u_x \\
\phi_{14} &= u_xu_y \\
\phi_{15} &= u_x^2 + u_y^2 - 2 \\
\phi_{16} &= -2u_y + (-1 + u_x^2)u_y + u_y(-1 + u_y^2)
\end{aligned}
\tag{63}
$$

The $O(\varepsilon_I^2)$ correction to the distribution function causes a change in the density at the surface, as well as the anisotropy on the mean square velocities. These functions have the form

$$
\begin{aligned}
\langle \rho(y) \rangle &= a_1(y)a_n + b_1(y)a_t + c_1(y)u_w^2 \,, \\
\langle \rho(u_x^2 - u_y^2) \rangle &= a_2(y)a_n + b_2(y)a_t + c_2(y)u_w^2 \,.
\end{aligned}
\tag{64}
$$

The above functions are shown as a function of distance from the surface y in Fig. 2. It is seen that the density near the surface increases as the parameter a_n is increased, which corresponds to decreasing the coefficient of restitution normal to the surface.

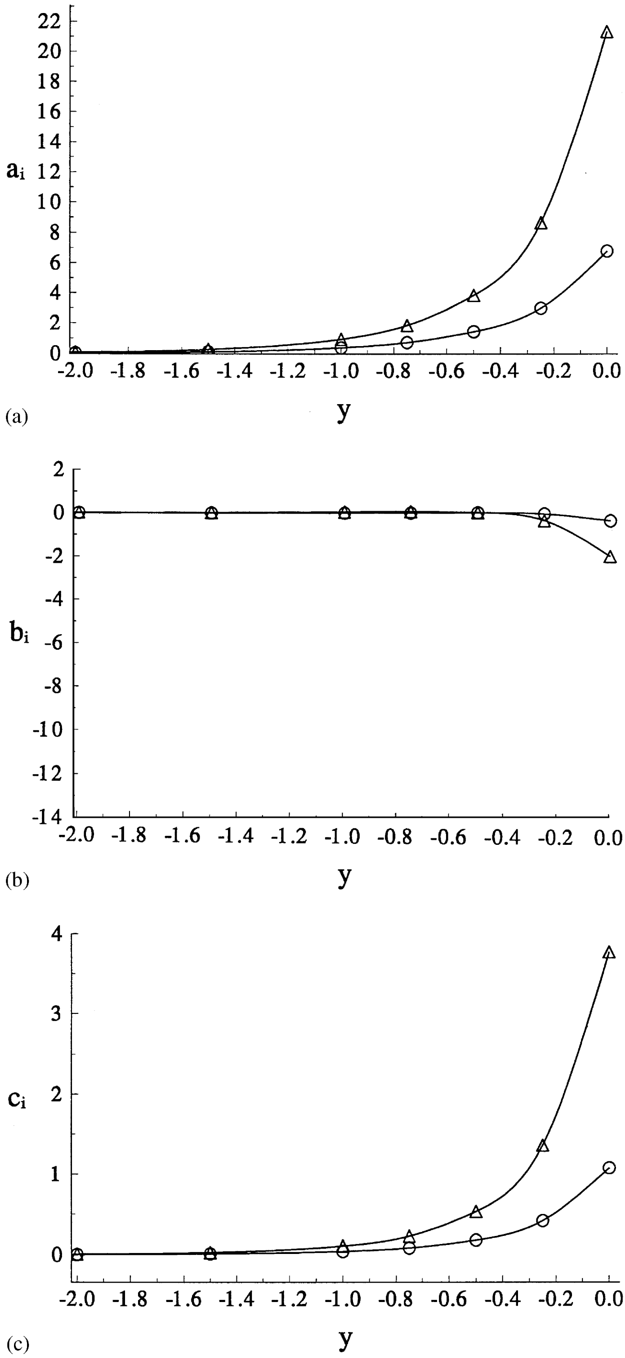

Fig. 2. The coefficients $a_1 - a_3$ (a), $b_1 - b_3$ (b) and $c_1 - c_3$ (c) in Eq. (45) for the $O(\varepsilon_I^2)$ corrections to the moments $\langle \rho \rangle$, $\langle \rho(u_x^2 - u_y^2) \rangle$, and $\langle \rho(u_x^2 + u_y^2) \rangle$ calculated using $N = 16$ as a function of distance from the wall for a finite shear flow.

5. Conclusions

The correction to the velocity distribution function for an inelastic granular material in shear flow has been analysed using a perturbation expansion of the Boltzmann equation in the small parameter $\varepsilon_I = (1-e)^{1/2}$, where e is the coefficient of restitution. The solution procedure involves an expansion of the correction to the distribution function in terms of a set of basis functions consisting of the eigenfunctions of the linearised Boltzmann collision operator. It is not possible to obtain analytical solutions for the eigenfunction of the linearised Boltzmann operator, and an expansion in a finite set of Hermite polynomials was used to obtain the eigenfunctions. It is expected that this approximation converges to the true eigenfunctions as the number of basis functions is increased. A systematic procedure is used to generate a hierarchy of equations for the higher-order corrections to the distribution function. In each of these equations, the linear operator acting on the unknown coefficients for the correction to the distribution function remains the same, while there is a change in the inhomogeneous term due to the lower-order corrections to the distribution function. In addition, the existence of a solution for the hierarchy of equations requires that the inhomogeneous term in the equation for the coefficient of an eigenfunction is zero if the corresponding eigenvalue is zero. This provides an expression for the 'granular temperature' as a function of shear rate.

Unlike the Chapman–Enskog procedure, the present analysis does not assume a form for the correction to the distribution function based on the symmetries of the inhomogeneous terms. Despite this, the present procedure has some advantages when compared to the Chapman–Enskog procedure. For homogeneous flows, an expansion in the eigenvalues of the linearised Boltzmann equation results in a set of independent linear algebraic equations for the coefficients in the expansion. In addition, the solvability conditions are easily determined. The results indicate that this procedure recovers the correct symmetries to a good approximation, and the errors due to the finite set of basis functions is numerically small even when the basis set consists of 16 eigenfunctions.

For the spatially varying granular medium, the present solution procedure is different from the classical solution of the linearised Boltzmann equation in a gas of elastic particles. In that case, the particle velocities are distributed according to the Maxwell–Boltzmann distribution at equilibrium, and the variation of the distribution function with time due to non-equilibrium effects is usually analysed [15]. The perturbations to the distribution function are expressed as an expansion in the eigenvalues of the linearised collision operator, and the decay rate of an eigenfunction is equal to the corresponding eigenvalue of the collision operator. In the present analysis, it is more appropriate to solve for the spatial variation of the distribution function in the gradient direction at steady state. This has been achieved by a normal form reduction of the equations for the perturbation to the distribution function, and analytical solutions have been obtained for the spatial variation of the normal modes. The results indicate that there are variations of $O(\varepsilon_I)$ in the mean velocity, and $O(\varepsilon_I^2)$ in the density and the mean square velocity near a solid surface due to wall effects, but these corrections decay over a distance of the order of the mean free path. The procedure used here differs from that of Jenkins and Richman [11] for the boundary conditions for an inelastic granular medium. In that case, the stress and energy flux in the flow were obtained assuming that the distribution function in the material is a Maxwell–Boltzmann distribution, and these were incorporated as boundary conditions in the mass and momentum equations for the granular flow. In the present case, boundary conditions for other eigenfunctions of the linearised Boltzmann operator are also satisfied by averaging over the microscopic reflection condition at the surface. This gives rise to some non-trivial effects such as the variation in the density and mean velocity near the surface.

Appendix A

The $\mathrm{O}(\varepsilon_I)$ and $\mathrm{O}(\varepsilon_I^2)$ corrections to the distribution function for $N = 16$ are

$$\Phi^{(1)} = 1.27337u_x u_y + 0.0447569u_x u_y(u_x^2 + u_y^2) + 0.002466u_x^2 u_y^2\,, \tag{A.1}$$

$$\Phi^{(2)} = 0.556424(1 + u_x^2 u_y^2) - 0.056424u_x^2 - 1.056424u_y^2 + 0.25u_x u_y\,. \tag{A.2}$$

The $\mathrm{O}(\varepsilon_I)$ and $\mathrm{O}(\varepsilon_I^2)$ corrections to the distribution function for $N = 36$ are

$$\begin{aligned}\Phi^{(1)} = {} & -1.37998u_x u_y + 0.0851323u_x u_y(u_x^2 + u_y^2) + 0.0003461u_x^3 u_y^3 \\ & + 0.00277314u_x u_y(u_x^4 + u_y^4) + 0.00960932u_x^3 u_y^3 - 0.00001387u_x^5 u_y^3\,,\end{aligned} \tag{A.3}$$

$$\begin{aligned}\Phi^{(2)} = {} & 0.843435 - 0.229389u_x^2 + 0.006884u_x^4 + 0.250000u_x u_y \\ & - 1.75955u_y^2 + 1.188233u_x^2 u_y^2 - 0.059337u_x^4 u_y^2 \\ & + 0.090042u_y^4 - 0.062311u_x^2 u_y^4 + 0.003493u_x^4 u_y^4\,.\end{aligned} \tag{A.4}$$

References

[1] S.B. Savage, Instability of an unbounded uniform granular shear flow, J. Fluid Mech. 241 (1992) 109.
[2] M. Babic, On the stability of rapid granular flows, J. Fluid Mech. 254 (1993) 127.
[3] M.A. Hopkins, M.Y. Louge, Inelastic microstructure in rapid granular flows of smooth disks, Phys. Fluids A 3 (1991) 47.
[4] S. McNamara, W.R. Young, Inelastic collapse and clumping in a one dimensional granular medium, Phys. Fluids A 5 (1993) 34.
[5] I. Goldhirsch, G. Zanetti, Clustering instability in dissipative gases, Phys. Rev. Lett. 50 (1993) 1619.
[6] R. Jackson, Hydrodynamic Stability of Fluid–Particle Systems in Fluidisation, Academic Press, London, 1985.
[7] J.T. Jenkins, S.B. Savage, A theory for rapid flow of identical, smooth, nearly elastic, spherical particles, J. Fluid Mech. 130 (1983) 187.
[8] C.K.K. Lun, S.B. Savage, D.J. Jeffrey, N. Chepurniy, Kinetic theories of granular flow: inelastic particles in a Couette flow and slightly inelastic particles in a general flow field, J. Fluid Mech. 140 (1984) 223.
[9] J.T. Jenkins, M.W. Richman, 'Grad's 13 moment system for a dense gas of inelastic particles, Arch. Rat. Mech. Anal. 87 (1985) 355.
[10] P.K. Haff, Grain flow as a fluid mechanical phenomenon, J. Fluid Mech. 134 (1983) 401.
[11] J.T. Jenkins, M.W. Richman, Boundary conditions for plane flows of smooth, nearly elastic, circular disks, J. Fluid Mech. 171 (1986) 53.

[12] N. Sela, I. Goldhirsch, S.H. Noskowicz, Kinetic theoretical study of a simply sheared two dimensional granular gas to Burnett order, Phys. Fluids 8 (1996) 2337.
[13] N. Sela, I. Goldhirsch, Hydrodynamic equations for rapid flows of smooth inelastic spheres, to Burnett order, J. Fluid Mech. 361 (1998) 41.
[14] C. Cercignani, The Boltzmann Equation and Its Applications, Springer, Berlin, 1975.
[15] P. Resibois, M. de Leener, Classical Kinetic Theory of Fluids, Wiley, New York, 1977.