## Velocity Autocorrelations and Viscosity Renormalization in Sheared Granular Flows

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The decay of the velocity autocorrelation function in a sheared granular flow is analyzed in the limit where the wavelength of fluctuations is larger than the "conduction length," so that energy is a nonconserved variable. The decay of the velocity autocorrelation function is much faster than that in a fluid at equilibrium for which energy is a conserved variable. Specifically, the autocorrelation function in a sheared granular flow decays proportional to  $t^{-3}$  in 2D and  $t^{-9/2}$  in 3D, in contrast with the decay proportional to  $t^{-1}$  in 2D and  $t^{-3/2}$  in 3D for a fluid at equilibrium. The renormalization of the viscosity due to mode coupling is evaluated using this form of the decay of the autocorrelation function. It is found that the logarithmic divergence in the viscosity in 2D, and the divergence of the Burnett coefficients in 3D, which is characteristic of a fluid of elastic particles at equilibrium, is absent in a sheared granular flow.

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Constitutive models have been developed for granular flows using methods similar to those used in the kinetic theory of gases [1-6]. An important assumption made in these kinetic theory calculations is the molecular chaos approximation, which neglects the effect of correlated collisions between particles. Because of this, these theories are generally assumed to be valid only in the dilute limit. It is well known that correlated collisions cause a significant change in the form of the constitutive relations for fluids at equilibrium. The seminal studies of Kawasaki and Gunton [7] and Yamada and Kawasaki [8] using mode coupling theory, Ernst and Dorfman [9] and Ernst et al. [10] using the ring kinetic theory, and Lutsko and Dufty [11] using a generalized Langevin formulation, showed that the shear viscosity in a fluid of elastic particles is a nonanalytic function of the strain rate. In two dimensions, the shear viscosity has the form  $\eta = \eta_0 + \eta' \log(\dot{\gamma})$ , while in three dimensions the shear viscosity has the form  $\eta =$  $\eta_0 + \eta' |\dot{\gamma}|^{1/2}$ , where  $\eta_0$  is the bare shear viscosity for a Newtonian fluid and  $\dot{\gamma}$  is the strain rate. This implies that the coefficient of viscosity diverges in a twodimensional fluid, while the Burnett coefficients diverge in a three-dimensional fluid. It is well known that the viscosity renormalization is caused by the long time tails in the velocity autocorrelation functions [10,12], where the autocorrelation functions decay as a power law  $t^{-d/2}$  in the long time limit, where *d* is the dimension of the system.

In the modeling of granular flows, it is important to know whether similar difficulties exist when the effects of correlated collisions are incorporated. A fundamental difference between a fluid at equilibrium and granular material is that energy is not conserved in the rapid flow of inelastic particles due to the dissipation of energy in collisions. In a granular shear flow, the energy dissipated in collisions is compensated by the production of energy due to the mean shear. Therefore, a hydrodynamic description of a granular material would include only the mass and momentum as conserved variables, while the "granular temperature" is determined by a balance between the rate of production due to the mean shear and the rate of dissipation due to inelastic collisions. The purpose of this analysis is to examine whether the long time tail in the velocity autocorrelation function and the nonanalyticity of the transport coefficients are encountered in this case as well. It should be noted that the results of this analysis are applicable to driven systems where energy is dissipated, but where *momentum* is conserved. In granular materials, for example, there is no force acting on particles between successive collisions thereby conserving momentum, and momentum is conserved in a collision. There are other driven dissipative systems where momentum is not conserved. An example is a thermostatted system, where a force is exerted on every particle in the direction opposite to the velocity direction. In this case, the momentum of the particle is not conserved, and the scaling or the growth rates of the hydrodynamic modes with wave number is different [13].

It should be noted that the calculations of Ernst and Dorfman [9] and Ernst et al. [10] are carried out with the linear shear flow as the base state, and this shearing results in heating due to viscous dissipation. However, as argued by Ernst et al. [10], the rate of viscous heating is proportional to  $\dot{\gamma}^2$ , whereas the viscous stresses are proportional to  $\dot{\gamma}$ , where  $\dot{\gamma}$ , the strain rate, is the small parameter in the analysis. Therefore, the viscous heating is neglected in a linear hydrodynamic analysis. In addition, the time scale required for the relaxation of the stresses is  $O(\dot{\gamma}^{-1})$ , and the correction to the temperature over the relaxation time for the stresses is  $O(\dot{\gamma}^{-1} \times \dot{\gamma}^2) \sim O(\dot{\gamma})$ , which is small in the limit of small strain rate. On the basis of these arguments, Ernst et al. [10] concluded that viscous heating is not significant over the time scales required for stress relaxation, and the temperature can be assumed to be a constant over these time scales. The problem of viscous heating does not arise for the shear flow of inelastic particles at steady state, because the viscous production of fluctuating energy is balanced by the dissipation due to inelastic collisions.

There are two components in the calculation. The first component is the calculation of the hydrodynamic modes in a sheared granular flow. These turn out to be very different from those for an elastic fluid at equilibrium in situations where energy is a nonconserved variable. The scalings of the hydrodynamic modes with wave number have already been obtained previously [14], and so they are only briefly summarized here. The second part of the calculation is to examine the effect of the modification of the hydrodynamic spectrum on the renormalization of the shear viscosity and the long time tails in the velocity autocorrelation function.

In a fluid of elastic particles, in the absence of shear, there are five conserved (slow) variables, the mass, three components of the momentum, and the energy, since these are conserved in interparticle collisions. Perturbations to these conserved variables decay as

$$\frac{\partial \phi_i}{\partial t} = -s_i(\mathbf{k})\phi_i,\tag{1}$$

where  $\phi_i$  is the conserved variable, and  $s_i(\mathbf{k})$  is the decay rate which tends to zero in the limit  $k \to 0$ . Of the five conserved variables, the two transverse momenta and the total energy are diffusive, with decay rate  $-\nu k^2$  and  $-D_t k^2$ , respectively, where k is the magnitude of the wave vector,  $\nu$  is the kinematic viscosity which is the ratio of the shear viscosity and density, and  $D_t$  is the thermal diffusivity. The density and longitudinal momentum (along the wave vector) are propagating modes, with growth rate proportional to  $\pm \iota c_s k - \Gamma k^2$ , where  $c_s$  is the speed of sound, and  $\Gamma$  is the rate of attenuation of the sound waves.

In the presence of mean shear, Eq. (1) is modified,

$$(\partial_t + \dot{\gamma}k_x \frac{\partial}{\partial k_y})\phi_i = s(\mathbf{k})\phi_i, \qquad (2)$$

where  $\dot{\gamma}$  is the strain rate, the mean velocity is in the *x* direction, the velocity gradient is in the *y* direction, and  $k_x$  and  $k_y$  are the components of the wave vector in the *x* and *y* directions, respectively. The above equation can be solved analytically to obtain

$$\phi_i(t) = \phi_i(0) \exp[-\nu_0 t (k^2 - \dot{\gamma} t k_x k_y + \frac{1}{3} \dot{\gamma}^2 t^2 k_x^2)]. \quad (3)$$

The decay of the hydrodynamic modes in a granular material is qualitatively different, because energy is not a conserved variable in a fluid of inelastic particles. It can be shown that a granular material contains an additional length scale, the "conduction length,"  $(\lambda/\epsilon)$ , intermediate between the mean free path  $\lambda$  and the macroscopic scale, where  $\epsilon = (1 - e)^{1/2}$ , and the coefficient of restitution *e* of the particles is close to 1. This length scale can be derived as follows. The energy dissipated in a collision between two particles is O((1 - e)T), where the "granular temperature" *T* is the mean square velocity of the particles (the

particle mass is considered to be 1 without loss of generality). The rate of diffusion of energy in the energy balance equation scales as  $O(D_T T/L^2)$ , where the thermal diffusivity  $D_T \sim \lambda T^{1/2}$  in kinetic theory of gases, while the rate of dissipation of energy is  $O((1-e)T^{3/2}/\lambda)$ . Equating these two terms, it is clear that the rate of diffusion and rate of dissipation are of equal magnitude for  $L_c \sim (\lambda/\epsilon)$ . For  $L \ll (\lambda/\epsilon)$  [or wave number  $k \gg (\epsilon/\lambda)$ ], energy is considered to be a conserved variable, and the spectrum of hydrodynamic modes is identical to that for a gas of elastic particles.

For  $L \gg (\lambda/\epsilon)$  [or wave number  $k \ll (\epsilon/\lambda)$ ], energy is not conserved, and the temperature is determined by a local balance between the rate of production of energy due to mean shear and the rate of dissipation due to inelastic collisions. The nature of the hydrodynamic modes is very different in this case, because energy is a nonconserved variable, and the only conserved variables are the mass and momenta. The linear growth rates of the hydrodynamic modes were calculated by the author [14], and we use these results to examine the long time tails in the velocity autocorrelation function. A distinction is made, once again, between the short time exponential growth or decay proportional to  $\exp(s(\mathbf{k})t)$  for  $\dot{\gamma}t \ll 1$ , equivalent to Eq. (1), and the long time behavior for  $\dot{\gamma}t \gg 1$ , when the turning of the wave vector due to the mean shear is incorporated, as in Eq. (3). In the short time limit, there are three solutions for the growth rate corresponding to the three conserved variables, which are the mass and two components of the momenta. The leading order solutions for the growth rate in the small wave-number limit are

$$s^{3} = \left(\frac{\dot{\gamma}^{3}\lambda^{2}}{\epsilon^{2}}[B_{1}(\phi)k_{x}k_{y} + B_{2}(\phi)\epsilon k_{x}^{2}]\right), \qquad (4)$$

where  $\lambda$  is the mean free path and  $B_1$  and  $B_2$  are dimensionless functions of the volume fraction, which have been explicitly calculated [14], and which approach finite values on the dilute limit. There are three solutions for the growth rate in Eq. (4), and the growth rates are proportional to  $k^{2/3}$  in the short time limit. In addition, depending on the signs of  $B_1$  and  $B_2$  and the magnitudes of  $k_x$  and  $k_y$ , either one or two of the solutions are linearly unstable in the short time limit. However, this short time instability is dampened in the long time limit  $\dot{\gamma}t \gg 1$ , by the turning of the wave vector due to mean shear, which results in a decay proportional to  $\exp(-\nu_0^2\dot{\gamma}^3k_x^2t^3/3)$  from Eq. (3).

The origin of the nonanalytic behavior of the shear viscosity in elastic fluids can be inferred using the ring kinetic equation for the pair distribution function in the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy, or the mode-coupling theory for the nonlinear coupling between the shear and longitudinal modes, or the Navier-Stokes-Langevin equations for mass, momentum, and energy conservation. All of these approaches provide identical results for the renormalization of the shear viscosity due to nonlinear interactions,

$$\eta - \eta_0 = \frac{\rho^2}{TV} \int_{\mathbf{k}'} \int_{\mathbf{k}''} \int_0^\infty dt \langle u_x(\mathbf{k}', t) u_y(-\mathbf{k}', t) u_x(\mathbf{k}'', 0) u_y(-\mathbf{k}'', 0) \rangle, \tag{5}$$

where *T* is the temperature in energy units, *V* is the volume,  $\rho$  is the density,  $u_x(\mathbf{k}, t)$  and  $u_y(\mathbf{k}, t)$  are the Fourier transforms of the components of the velocity, and  $\int_{\mathbf{k}} \equiv (2\pi)^{-d} \int d\mathbf{k}$ , where *d* is the dimension of the system. Throughout the analysis, we set the particle mass equal to 1 without loss of generality, so that all mass dimensions are scaled by the particle mass. In Eq. (5), the averaging is carried out over the velocity fluctuations in a linear shear flow. An identical equation is also obtained starting from the Green-Kubo formula for the shear viscosity in terms of the stress-stress correlation, though the ensemble average is carried out over equilibrium fluctuations in that case.

The average in Eq. (5) has contributions from both the longitudinal and the transverse shear modes. The contribution from the transverse modes can be obtained, to within a multiplicative constant, as follows. Equation (1) can be used for the decay rate of perturbations to the conserved variables in the linear approximation, and the evolution of the velocity autocorrelation is of the form

$$\langle u_i(\mathbf{k}, t)u_j(-\mathbf{k}, 0)\rangle = \exp(s(\mathbf{k})t)\langle u_i(\mathbf{k}, 0)u_j(-\mathbf{k}, 0)\rangle.$$
(6)

When this is inserted into the equation for the viscosity (5), we obtain

$$\eta - \eta_0 = \frac{\rho^2}{TV} \\ \times \int_0^\infty dt \int_{\mathbf{k}} \langle u_x(\mathbf{k}, 0) u_x(-\mathbf{k}, 0) u_y(\mathbf{k}, 0) u_y(-\mathbf{k}, 0) \rangle \\ \times \exp(2s(\mathbf{k})t).$$
(7)

Since the equal time average  $\langle u_i(\mathbf{k}, 0)u_j(-\mathbf{k}', 0)\rangle$  is equal to  $(T/\rho)(2\pi)^d \delta(\mathbf{k} + \mathbf{k}')\delta_{ij}$  at equilibrium, the expression for the viscosity is

$$\eta = T(2\pi)^d \int_0^\infty dt \int_{\mathbf{k}} \exp(2s(\mathbf{k})t)$$
(8)

$$=T\int_{0}^{\infty} dt \frac{1}{2\nu_0 t}$$
 in 2D, (9)

$$= T \int_0^\infty dt \frac{1}{(2\nu_0 t)^{3/2}} \quad \text{in 3D,} \tag{10}$$

where we have used the substitution  $s(\mathbf{k}) = -\nu_0 k^2$ . Equations (9) and (10) indicate the presence of "long time tails" in the velocity autocorrelation function, where the autocorrelation function decays as  $t^{-d/2}$  instead of the exponential decay expected in single relaxation time processes. In two dimensions, the above time integral is divergent, resulting in a divergence in the viscosity. In three dimensions, the integral is convergent, but decays slowly as  $t^{-1/2}$  in the long time limit. In order to obtain the nonanalytic form of the viscosity as a function of strain rate, it is necessary to incorporate an additional effect in the evolution equation for the viscosity, which is the effect of mean shear on the evolution equation for the velocity fluctuations in (1). Note that the long time tail in the velocity autocorrelation function is an equilibrium property of the system, and is not caused by shear.

It is easy to see that the fast decay of the velocity perturbation proportional to  $\exp(-\nu_0 \dot{\gamma}^2 t^3 k_x^2/3)$  due to shear in Eq. (3) cuts off the time integrals in Eqs. (9) and (10) at an upper limit proportional to  $\dot{\gamma}^{-1}$ , thus giving an expression for the viscosity.

$$\eta - \eta_0 = \eta_c + \frac{A_{2t}T}{\nu_0} \log(\dot{\gamma})$$
 in 2D, (11)

$$= \eta_c + \frac{A_{3t}T}{\nu_0^{3/2}} \dot{\gamma}^{1/2} \quad \text{in 3D,} \tag{12}$$

where  $A_{2t}$  and  $A_{3t}$  are dimensionless constants in two and three dimensions, respectively, and the contribution  $\eta_c$  in (11) and (12) results from the lower cutoff for the time integration, since the integrals are divergent at t = 0. In addition to the contribution due to the coupling of the shear modes, there is an additional contribution due to the coupling of the longitudinal modes, and so the final expression for the renormalization of the viscosity is

$$\eta - \eta_0 = T \left( \frac{A_{2t}}{\nu_0} + \frac{A_{2l}}{\Gamma_0} \right) \log(\dot{\gamma}) \quad \text{in 2D,} \tag{13}$$

$$= T \left( \frac{A_{3t}}{\nu_0^{3/2}} + \frac{A_{3l}}{\Gamma_0^{3/2}} \right) \dot{\gamma}^{1/2} \quad \text{in 3D,} \qquad (14)$$

where  $\Gamma_0$  is the bare damping coefficient for the longitudinal modes, and  $A_{2l}$  and  $A_{3l}$  are dimensionless constants. Equations (13) and (14) are identical in form to the results of Ernst *et al.* [10] and Lutsko and Dufty [11], though substantially more work is required to evaluate the constants in the solutions (13) and (14).

For a fluid of inelastic particles, the viscosity renormalization due to mode coupling can be determined using Eq. (5), in which the growth rate of the hydrodynamic modes is given by Eq. (4). Since the growth rate is proportional to  $(k\lambda/\epsilon)^{2/3}\dot{\gamma}$ , the integral over the wave number in Eq. (8) provides the following time dependence for the renormalized viscosity,

$$\eta - \eta_0 = C_2 T \int_0^\infty dt \frac{\epsilon^2}{(\dot{\gamma}t)^3 \lambda^2} \quad \text{in 2D,} \qquad (15)$$

$$= C_3 T \int_0^\infty dt \frac{\epsilon^3}{(\dot{\gamma}t)^{9/2} \lambda^3} \quad \text{in 3D,} \qquad (16)$$

where  $C_2$  and  $C_3$  are constants resulting from the numerical integration over the wave vector space, and we have not explicitly written down the contributions  $\eta_c$  resulting from the upper cutoff in the wave number as in Eqs. (11) and (12), since these could be incorporated in the definition of the bare viscosity. Equations (15) and (16) indicate that the long time behavior of the velocity autocorrelation functions in a granular flow are also very different from those in a fluid at equilibrium, due to the lack of conservation of energy. In two dimensions, the velocity autocorrelation function decays proportional to  $t^{-3}$ , in contrast to the  $t^{-1}$  decay in an elastic fluid at equilibrium. In three dimensions, the decay is proportional to  $t^{-9/2}$ , in contrast to the decay proportional to  $t^{-3/2}$  in an elastic fluid at equilibrium.

The time integrals in Eqs. (15) and (16) are easily evaluated to within a multiplicative constant, using an upper cutoff at  $t \propto \dot{\gamma}^{-1}$  as before. The final result for the viscosity renormalisation in two dimensions is of the form

$$\eta - \eta_0 \propto \frac{T\epsilon^2}{\dot{\gamma}\lambda^2} \propto \dot{\gamma} \tag{17}$$

since  $T \propto (\dot{\gamma}^2 \lambda^2 / \epsilon^2)$ . The leading order contribution to the viscosity in two dimensions,  $\eta_0$ , is proportional to  $T^{1/2}/\sigma \sim (\dot{\gamma}\lambda/\epsilon\sigma)$ . Therefore, the viscosity renormalization is proportional to  $\epsilon(\sigma/\lambda)$  times the leading order viscosity. This renormalization is of the same order in the  $\epsilon$  expansion as the Burnett correction, and therefore it can be concluded that correlated collisions do not lead to divergent viscosities of Burnett coefficients even in two dimensions. A similar calculation in three dimensions provides the result,

$$\eta - \eta_0 \propto \frac{\epsilon \dot{\gamma}}{\lambda}.$$
 (18)

Thus, the viscosity renormalization in three dimensions is  $\epsilon^2 (\sigma/\lambda)^2$  smaller than the leading order viscosity. The  $O(\epsilon^2)$  correction to the viscosity corresponds to the super-Burnett terms in the  $\epsilon$  expansion, and therefore both the shear viscosity and the Burnett corrections are not divergent in three dimensions.

The transition between the elastic fluid limit (where energy is conserved) and the present case of a granular material (where energy is not conserved) can be understood on the basis of the three lengths scales in the system, the macroscopic scale *L*, the microscopic scale (mean free path)  $\lambda$ , and this intermediate scale  $L_c = \lambda/(1-e)^{1/2}$ .

Since energy is conserved for  $L \ll L_c$ , the dynamics of an elastic fluid is recovered. For  $L \gg L_c$ , the dynamics of an inelastic sheared fluid discussed here is obtained. As the inelasticity is decreased  $(e \rightarrow 1)$  at constant system size L, the conduction length  $L_c$  diverges, and the elastic limit is recovered when  $L_c$  becomes greater than the system size L.

Thus, the present analysis establishes that long time tails in the velocity autocorrelation function, and the presence of a divergent viscosity in 2D and divergent Burnett coefficients in 3D, are not encountered in sheared granular flows where energy is a nonconserved variable. This indicates that the form of the constitutive relations determined by the Chapman-Enskog procedure, which neglects correlations, is valid for granular flows, and correlations do not cause the divergences in the viscosity or Burnett coefficients encountered in elastic fluids at equilibrium. However, there will be numerical contributions due to the upper wave-number cutoff in Eq. (8), and the lower time cutoff in Eqs. (15) and (16), which could alter the numerical values of the transport coefficients. Work is currently in progress to evaluate these numerical contributions.

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