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Kinetic theory for the density plateau in the granular flow down an inclined plane

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Abstract. – The striking lack of observable variation of the volume fraction with height in the center of a granular flow down an inclined plane is analysed using constitutive relations obtained from kinetic theory. It is shown that the rate of conduction in the granular energy balance equation is $O(\delta^2)$ smaller than the rate of production of energy due to mean shear and the rate of dissipation due to inelastic collisions, where the small parameter $\delta = (d/(1-e_n)^{1/2}H)$, d is the particle diameter, e_n is the normal coefficient of restitution and H is the thickness of the flowing layer. This implies that the volume fraction is a constant in the leading approximation in an asymptotic analysis in small δ. Numerical estimates of both the parameter δ and its pre-factor are obtained to show that the lack of observable variation of the volume fraction with height can be explained by constitutive relations obtained from kinetic theory.

The flow of a granular material down an inclined plane has been studied extensively using computer simulations [1–3]. One of the most remarkable features of these simulation results is that the volume fraction of the particles is a constant in the bulk of the flow, the granular temperature and all stress components are linear functions of height, and the mean velocity increases as the square root of the height from the bottom of the layer. The volume fraction is found to be, within numerical accuracy, independent of position in the flow (apart from two thin layers at the top and bottom of thickness about 3-5 particle diameters where the volume fraction varies with position), independent of the total height of the material, and dependent only on the angle of inclination of the inclined plane. There have been several studies of this flow, both phenomenological and kinetic theory based [4–7], but a clear explanation for the lack of observable variation in the volume fraction is still not available.

Kinetic theories for granular materials exploit the analogy between the motion of discrete particles in the granular material and the motion of molecules in a molecular gas. These include approximate approaches that modified the Navier-Stokes equations by adding a dissipation term due to inelastic collisions in the energy equation [8–11], as well as asymptotic approaches that used expansions in the inelasticity and the Knudsen number [12–15]. From dimensional analysis it can be inferred that the stress is equal to the square of the strain rate

times the particle diameter times a function of the volume fraction, since there are no other time scales in the problem, and this type of relationship is called the "Bagnold law".

It is usually assumed that these theories are applicable only in the dilute limit, but simulations [1] have shown that the Bagnold law for the stress tensor applies even for dense granular flows with volume fraction ranging from 0.55 to about 0.6. The validity of the Bagnold law indicates that the inverse of the strain rate is the only time scale in the problem, and that there is no interference between the flow time (inverse of strain rate) and the time between collisions. Additional support for this conclusion is provided by the simulations of Mitarai and Nakanishi [3], who found that the constitutive relations based on kinetic theory could provide a fairly accurate prediction of the stress tensor in this flow, though the rate of dissipation of energy is not accurately predicted. In addition, it is known [14, 15] that if the rate of conduction of energy is neglected in the energy balance equation, kinetic theory predicts that the density is independent of height in the bulk of the flow. In the present analysis, we determine the effect of conduction on the density profile in the bulk of a steady granular flow down an inclined plane.

The granular material is composed of hard-sphere particles of diameter d flowing down a plane inclined at an angle θ to the horizontal. A Cartesian coordinate system is used, where the velocity and velocity gradient are in the x and y directions, respectively. The mass of a particle is set equal to 1 for simplicity. The shear and normal stress balances, and the constant ratio of the shear and normal stresses, are

$$\begin{aligned} (\mathrm{d}\sigma_{xy}/\mathrm{d}y) &= -\rho g \sin(\theta), \\ (\mathrm{d}\sigma_{yy}/\mathrm{d}y) &= \rho g \cos(\theta), \\ (\sigma_{xy}/\sigma_{yy}) &= -\tan(\theta). \end{aligned} \tag{1}$$

The energy equation at steady state is

$$\frac{\mathrm{d}}{\mathrm{d}y} K \frac{\mathrm{d}T}{\mathrm{d}y} + \mu\dot{\gamma}^2 - D = 0, \tag{2}$$

where K is the thermal conductivity, D is the rate of dissipation of energy, T is the "granular temperature", μ is the viscosity and $\dot{\gamma}$ is the strain rate.

The expressions for the shear and normal stresses depend on the approximation used for the stress tensor. In the Burnett approximation, the shear and normal stresses are given by

$$\begin{aligned} \sigma_{xy} &= \mu\dot{\gamma}, \\ \sigma_{yy} &= -p + B\dot{\gamma}^2. \end{aligned} \tag{3}$$

In the Navier-Stokes approximation, the Burnett coefficient B is set equal to zero. The results of the present analysis indicate that the variation of density with height in the Navier-Stokes and Burnett approximations are qualitatively the same, though an earlier study [15] has shown that there is a significant variation in the dependence of density on the angle of inclination. Consequently, for the present purposes, we use the Navier-Stokes approximation for the shear stress with B set equal to zero.

It is convenient to express the viscometric coefficients and the dissipation coefficient as a product of two functions, one of which is a dimensionless function of volume fraction, and the other is a product of suitably chosen powers of the granular temperature and particle diameter, the latter having the same dimensions as the viscometric function under consideration. (Note that the granular temperature has dimensions of the square of the velocity, since the mass is set equal to 1.) From dimensional analysis, it can be inferred that $p = p_\phi(T/d^3)$ $K = K_\phi(T^{1/2}/d^2)$, $\mu = \mu_\phi(T^{1/2}/d^2)$, and $D = D_\phi\varepsilon^2(T^{3/2}/d^4)$, where the variables with subscript

$_\phi$ are dimensionless functions of the volume fraction, and $\varepsilon = (1-e_n)^{1/2}$ is the small parameter used in the expansion [14, 15] to determine the constitutive relations, where e_n is the normal coefficient of restitution. The parameter ε^2 is written separately in the expression for the rate of dissipation of energy in order to ensure that the rate of dissipation goes to zero in the limit of elastic collisions. The strain rate can be expressed in terms of the temperature using eqs. (3) for the stresses, and eqs. (1) for the ratio of the stresses,

$$\dot{\gamma} = \frac{\tan(\theta) p_\phi T^{1/2}}{\mu_\phi d} = G(\phi, \tan(\theta))(T^{1/2}/d). \tag{4}$$

It is convenient to scale the y coordinate by the height of the flowing layer, $y_* = (y/H)$, since this is the length scale for the variation of the potential energy of the particles in the momentum balance equation in the y-direction,

$$\frac{1}{H}\frac{\mathrm{d}(p_\phi T/d^3)}{\mathrm{d}y_*} = \rho g \cos(\theta) = (6/\pi d^3)\phi g \cos(\theta). \tag{5}$$

Equation (5) indicates that for the pressure to balance the weight per unit area of $\mathrm{O}(gH)$, the temperature has to scale as gH in the flowing layer. A scaled temperature can be defined as $T_* = (T/gH)$. This scaled temperature is inserted into the energy balance equation, eq. (2), and the resulting equation is divided by the coefficient of D_ϕ in the last term on the right side, to obtain

$$\delta^2 \frac{\mathrm{d}}{\mathrm{d}y_*}\left(K_\phi T_*^{1/2}\frac{\mathrm{d}T_*}{\mathrm{d}y_*}\right) = -(\mu_\phi G_*(\phi, \tan(\theta))^2 - D_\phi)T_*^{3/2}, \tag{6}$$

where $\delta = (d/\varepsilon H)$ is the ratio of the "conduction length" [14, 15] and the height H, $G_*(\phi, \tan(\theta)) = (G(\phi, \tan(\theta))/\varepsilon)$, and $G(\phi, \tan(\theta))$ is defined in eq. (4).

In eq. (6), it is apparent that the parameter multiplying the conduction term on the left side is small if the height is large compared to (d/ε), or $H \gg (d/\varepsilon)$. This condition is satisfied for the chute flows of Silbert *et al.* [1]. For example, for $H = 40d$, the parameter δ varies from about 0.08 for $e_n = 0.9$ to about 0.0353 for $e_n = 0.5$. In this case, an asymptotic expansion can be employed and the density and temperature can be expanded in the small parameter δ, $\phi = \phi^{(0)} + \delta\phi^{(1)} + \delta^2\phi^{(2)}$, and similar expansions for T and $\dot{\gamma}$. When these expansions are inserted in the energy equation (6), the leading-order equation is

$$\mu_\phi^{(0)}(G_*^{(0)})^2 - D_\phi^{(0)} = 0, \tag{7}$$

where we use the notation $\star^{(0)} = \star(\phi)|_{\phi=\phi^{(0)}}$ for the viscosity, thermal conductivity, pressure and the rate of dissipation of energy. In the above expression, the left side is a function of the density $\phi^{(0)}$ which is, in general, a function of y, whereas the right side is independent of height. Therefore, the equality in eq. (7) can be valid at all values of y only if the leading solution for the volume fraction $\phi^{(0)}$ is independent of y. This density can be explicitly determined as a function of angle θ from a knowledge of the functional forms of $D_\phi^{(0)}$, $\mu_\phi^{(0)}$ and $p_\phi^{(0)}$, as was carried out in an earlier study [15]. Once the value of $\phi^{(0)}$ is known, the leading-order temperature field can be determined from the momentum balance equation,

$$T_*^{(0)} = \frac{6\phi^{(0)}(1-y_*)\cos(\theta)}{\pi p_\phi^{(0)}}. \tag{8}$$

Here, we have used the condition that $T_*^{(0)} = 0$ at the free surface $y_* = (y/H) = 1$ in order to fix the constant in the above equation. Note that it is not possible to apply boundary conditions for the temperature field, since we have neglected the conduction of energy in the leading approximation in eq. (6), and converted the equation from a second-order differential equation to a zeroth order differential equation. The conduction term has to be included in thin layers at the top and bottom using a theory similar the boundary layer theory for viscous flows, and work is currently in progress to analyse these layers.

Next, we turn to the calculation of the higher-order corrections to the volume fraction due to the gradient term in the left side of eq. (6). Since the inhomogeneous term is $O(\delta^2)$, the first correction to the density $\phi^{(1)}$ is identically equal to zero. The second correction to the density is obtained by substituting the density and temperature expansion into eq. (6), and retaining all terms of $O(\delta^2)$,

$$\frac{\mathrm{d}}{\mathrm{d}y_*}\left(K^{(0)}{}_\phi {T^{(0)}}_*^{1/2}\frac{\mathrm{d}{T^{(0)}}_*}{\mathrm{d}y_*}\right) = -\left.(\mu_\phi G_*^2 - D_\phi)T_*^{3/2}\right|_2, \tag{9}$$

where $K_\phi^{(0)} = K_\phi(\phi^{(0)})$, and the superscript $_2$ refers to the $O(\delta^2)$ contribution to the terms on the right. Since the density-dependent term in brackets on the right side of eq. (9) is zero in the leading approximation, and the first corrections to ϕ and T_* are zero, and the $O(\delta^2)$ contribution on the right side is given by

$$\left.(\mu_\phi G_*^2 - D_\phi)T_*^{3/2}\right|_2 = \left.\frac{\mathrm{d}}{\mathrm{d}\phi}(\mu_\phi G_*^2 - D_\phi)\right|_{\phi=\phi^{(0)}} \phi^{(2)}{T^{(0)}}_*^{3/2}. \tag{10}$$

Using the above expression for the right side of eq. (9), and inserting the leading-order solution for $T_*^{(0)}$, we obtain

$$\phi^{(2)} = -\frac{K_\phi^{(0)}}{2(1-y_*)^2}\left(\left.\frac{\mathrm{d}}{\mathrm{d}\phi}(\mu_\phi G_*^2 - D_\phi)\right|_{\phi=\phi^{(0)}}\right)^{-1}. \tag{11}$$

The numerical estimate or $\phi^{(2)}$ is obtained using the constitutive relations for smooth nearly elastic model [14] (where the post-collisional relative velocity along the line joining centers is $-e_n$ times the pre-collisional value), and the rough nearly elastic model [15] (where the post-collisional relative velocity parallel and perpendicular to the line joining particle centers are $-e_n$ and $-e_t$ times their pre-collisional values, respectively). Here, e_n is the normal coefficient of restitution which varies between 0 and 1, and $e_n = 1$ corresponds to perfectly elastic collisions where energy is conserved. For rough particles, $e_t = -1$ corresponds to smooth particles where there is no change in the relative velocity perpendicular to the line joining centers in a collision, while $e_t = 1$ corresponds to perfectly rough particles where the relative velocity perpendicular to the line joining centers is reversed in the collision. The constitutive relations for the stress were obtained using an expansion in the parameter in the expansion is defined as $\varepsilon = (1 - e_n)^{1/2}$. In the case of rough particles, the parameter $a_t = (1 - e_t^2)/(1 - e_n^2)$ is a numerical O(1) factor. The results for the viscometric coefficients in the two limits are shown in table I. Numerical results are obtained for two different forms of the pair distribution function for hard-sphere fluids, the Carnahan-Starling pair distribution function

$$\chi(\phi) = (2 - \phi)/(2(1 - \phi)^3), \tag{12}$$

and the high-density pair distribution function,

$$\chi(\phi) = (1 - (\phi/\phi_c)^{1/3})^{-1}, \tag{13}$$

TABLE I – *Viscometric coefficients obtained from kinetic theory for smooth nearly elastic particles* [14] *and rough nearly elastic particles* [15].

	Smooth	Rough
p_ϕ	$(6\phi/\pi)(1+2(2-\varepsilon^2)\phi\chi)$	$(6\phi/\pi)(1+2(2-\varepsilon^2)\phi\chi)$
μ_ϕ	$(0.176/\chi)+0.564\phi+2.175\phi^2\chi$	$(0.195/\chi)+0.892\phi+3.112\phi^2\chi$
D_ϕ	$(144/\pi^{3/2})\phi^2\chi$	$(144/\pi^{3/2})\phi^2\chi(1+a_t)$
K_ϕ	$(0.662/\chi)+3.174\phi+8.12\phi^2\chi$	$(1.014/\chi)+5.015\phi+19.27\phi^2\chi$
B	$(0.0410/\chi)+(0.0116/\phi\chi^2)$ $-0.00381\phi-0.136\chi\phi^2$	$(0.04094/\chi)+(0.00433/\phi\chi^2)$ $-0.191\phi-1.05\phi^2\chi$

where we assume that the density at close packing, ϕ_c, is 0.65. The Carnahan-Starling pair distribution function is accurate at low and moderate volume fractions, but does not show the expected divergence of the pair distribution function as the volume fraction for close packing is approached. The high-density pair distribution does diverge as the volume fraction for close packing is approached, but is not accurate in the low volume fraction limit. It is shown below in fig. 1 that results for the correction to the density are qualitatively the same for both forms of the pair distribution function, and their numerical values are also close to each other. Therefore, it is expected that a similar behaviour will also be observed for other forms of the pair distribution function, so long as the numerical value of the pair distribution function at contact becomes large compared to 1 in the close-packed limit.

The numerical estimates for the leading-order volume fraction $\phi^{(0)}$ as a function of the angle of inclination were evaluated earlier [15], and so we restrict attention to the variation of $(\phi^{(2)}(1-y_*)^2)$ as a function of $\phi^{(0)}$. Since the constitutive relations are evaluated correct to $\mathrm{O}(\varepsilon^2)$, and the terms on the right side of the conduction equation (6) is $\mathrm{O}(\varepsilon^2)$, the value of $\phi^{(2)}$ can be evaluated only to leading order in the limit of small ε. The numerical results

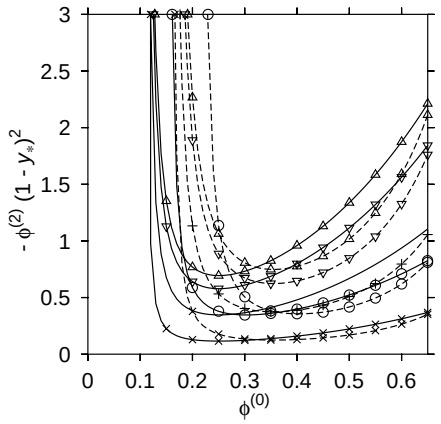

Fig. 1 – The parameter $(\phi^{(2)}(1-y_*)^2)$ (eq. (11)) as a function of $\phi^{(0)}$ for the smooth nearly elastic model ($\circ$); the rough nearly elastic model with $a_t = 0$ ($\triangle$); $a_t = 0.2$ ($\triangledown$); $a_t = 1.0$ ($+$); $a_t = 5.0$ ($\times$). The broken lines show the results when the pair distribution function is given by the Carnahan-Starling equation of state (eq. (12)), and the solid lines show the results when the pair distribution function is given by the high-density equation of state (eq. (13)).

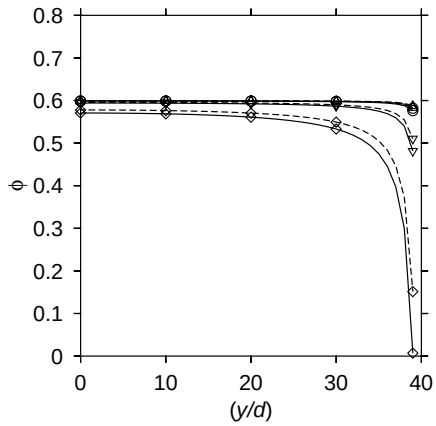

Fig. 2 – Volume fraction ϕ as a function of the ratio of the height and particle diameter (y/d) for the smooth nearly elastic model (dashed line); the rough nearly elastic model with $a_t = 1.0$ (solid line), for $H = 40d, e_n = 0.5$ ($\circ$), $H = 40d, e_n = 0.7$ ($\triangle$), $H = 40d, e_n = 0.9$ ($\bigtriangledown$) and $H = 40d, e_n = 0.98$ ($\diamond$). The pair distribution function was assumed to be the high-density equation of state (eq. (13)) in all cases.

for $(\phi^{(2)}(1 - y_*)^2)$ as a function of volume fraction are shown in fig. 1 for the smooth and rough nearly elastic particle models. It is observed that $(\phi^{(2)}(1 - y_*)^2)$ has a maximum value of about 2 near close packing for rough nearly elastic particles with $a_t = 0$, and is always lower than 2 for all other parameter values studied. The value of $\phi^{(2)}$ does increase at lower volume fractions and it diverges at a volume fraction close to 0.1 because the denominator (last term on the right side of eq. (11)) passes through zero, but such low volume fractions are not encountered in practical applications or in the simulations [1, 3]. This indicates that the variation in volume fraction is, at most, about 1.2% even for thin layers of thickness equal to 40 particle diameters and $e_n = 0.9$, for which $\delta^2 = 0.0064$, when the volume fraction is greater than about 0.3 in three dimensions, and the variation decreases as the collisions become more inelastic. These variations may be difficult to observe in graphs of simulation results, since they are smaller than the typical symbol size or error bar in the graph.

An example of the density profile predicted by the above analysis, for $\phi^{(0)} = 0.60$ and for different coefficients of restitution, is shown for a layer with thickness equal to 40 particle diameters in fig. 2. It is observed that the density profile is remarkably constant between $e_n = 0.5$ and $e_n = 0.9$, though this solution is not valid within boundary layers of thickness about 5 particle diameters at the top and bottom as noted earlier. The density profile does show some variation for $e_n = 0.98$ because the parameter δ is 0.177, and is no longer small, but even this variation is small in the center of the layer. The simulations of Silbert *et al.* [1] do show an example of a profile at $e_n = 0.98$ which looks much flatter than fig. 2, but it should be noted that this is in two dimensions, and Silbert *et al.* had a non-zero friction coefficient, which would result in a larger energy dissipation than that for frictionless particles with $e_n = 0.98$. Thus, the present analysis captures the remarkable lack of observable variation of the volume fraction with height, and with angle of inclination near close packing.

Lastly, we discuss in further detail the parameter δ, which is the ratio of two length scales (d/ε) and H. Consider a variation in temperature over a length scale L_c. The rate of conduction of energy is $(KT/L_c^2) = K_\phi(T^{1/2}/d^2)(T/L_c^2)$, while the rate of dissipation is $D = (D_\phi \varepsilon^2 T^{3/2}/d^4)$, where the expressions for K_ϕ and D_ϕ are provided in table I. The rate of

conduction and dissipation are of equal magnitude for $L_c = (K_\phi/D_\phi)^{1/2}(d/\varepsilon)$. For $H \gg L_c$, the rate of dissipation is large compared to the rate of conduction, and so the rate of conduction of energy (left side of eq. (6)) can be neglected in the leading approximation. However, there are regions of thickness L_c at the top and bottom of the flowing layer where the conduction of energy is important, and these have to be analysed using a boundary layer analysis. The ratio (d/ε) shows a very modest variation between about 1.5 and 3 particle diameters when the normal coefficient of restitution varies between 0.9 and 0.5, and this is consistent, up to a numerical factor of about 2, with the observation in simulations that the variation in density at the top and bottom is restricted to regions of thickness equal to 3–6 particle diameters. In the low-density limit, $(K_\phi/D_\phi) \propto \phi^{-2}$, and so the conduction length scales as $(d/\varepsilon\phi) \sim (1/\rho d^2\varepsilon) \sim (\lambda/\varepsilon)$, where λ is the mean free path in a dilute gas. Near close packing, both K_ϕ and D_ϕ increase proportional to the pair distribution function χ, and so $L_c \sim (d/\varepsilon)$.

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